SYSTEM MODELLING WITH REAL PARAMETRIC UNCERTAINTIES

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Abstract: This paper is about modelling linear systems with real parametric uncertainties. Each uncertain parameter is seen as a gain whose associated uncertainty is modelled by a normalized feedback relating fictitious variables. This methodology is systematized through the use of block diagrams built from four basic blocks implementing direct and inverse uncertain gains. This approach is specially useful for systems modelled from physical laws and can be applied to large systems.

Keywords: Parametric; Linear; Uncertainties; Robust.

1. INTRODUCTION

Robustness properties are an increasingly important aspect of modern controller synthesis. The goal of a robust control system is to maintain its properties despite the occurrence of changes in the system. These changes define a set of different possibilities for operation conditions. These can result from physical changes such as the replacement of a component for another not exactly equal, its slowly variation in time, its unknown exact value or even changes in the operating point. These considerations shows that models should incorporate the knowledge about possible modifications or variations of the system and be as accurate as possible.

Models based on real parameters, such as state space equations derived from physical laws, allow an accurate representation of systems. In present days, the physical laws that describe the behavior of a system are most often well known but the values of their coefficients are only known to a certain degree of accuracy. This is also true for parametric models modelling the spectrum of the type of disturbances affecting a system. This approach is in contrast to mathematically motivated parametrizations that can be found frequently in literature (e.g. norm bounded uncertainties of the matrices in state-space models, interval matrices, complex parameters, frequency domain uncertainties, overbounding by interval polynomials). (Ackermann, 1993, p.V)

Structured models, in principle, lead to less conservative controller designs, which could mean a reduction in controller complexity for the same

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achieved performance or better performance for the same controller complexity.

This paper is about system modelling with real parametric uncertainties. The structure of the paper is as follows. Section 2 presents the parametric model setup used and introduce the concept of “modelling capacity” together with examples. Section 3 presents the conclusions.

2. PARAMETRIC MODELS WITH REAL UNCERTAINTIES

Consider the control setup shown in figure 1, where \( w \in \mathbb{R}^{m_w} \) is a disturbance input (such as tracking signals, measurable and non measurable disturbances), \( u \in \mathbb{R}^{m_u} \) is the control input, \( z \in \mathbb{R}^{m_z} \) is the controlled output (such as error and control signals) and \( y \in \mathbb{R}^{m_y} \) is the measured output. The generalized plant \( G \) includes the system to be controlled and the additional elements to generate the signals involved. It is assumed that \( G \) does not have direct terms from \( w \) to \( z \) and from \( w \) to \( y \). The \( F \) block represents the controller.

The generalized system \( G \) admits the following state space representation

\[
G \rightarrow \begin{cases}
    \dot{x} = Ax + B_1 w + B_2 u \\
    z = C_1 x + D_{12} u \\
    y = C_2 x + D_{22} u
\end{cases}
\]  

(1)

for which one admits that any coefficient of any matrix of the state space model can be a function of several uncertain parameters.

2.1 Modelling interval uncertainties

Let \( k \in \mathbb{R} \) be an uncertain parameter. Its exact value is unknown but it is assumed that it belongs to a known interval \( k \in [k_{\min} ; k_{\max}] \). The most usual ways of modelling \( k \) are as an absolute tolerance,

\[
k = k_0 + \Delta d_k,
\]

(2)

with,

\[
k_0 = \frac{k_{\min} + k_{\max}}{2}, \quad d_k = \frac{k_{\max} - k_{\min}}{2}
\]

(3)

or as a relative tolerance,

\[
k = k_0(1 + \Delta p_k),
\]

(4)

with,

\[
k_0 = \frac{k_{\min} + k_{\max}}{2}, \quad p_k = \frac{k_{\max} - k_{\min}}{k_{\max} + k_{\min}}
\]

(5)

where \( \Delta \in \mathbb{R} \) and \( \Delta \in [-1; 1] \). The uncertainty in \( k \) stays linked to the normalized parameter \( \Delta \) in the unit interval. The conversion between the two representations is easily made by \( p_k = d_k/k_0 \).

In the field of dynamic systems, each uncertain parameter can be viewed as a gain modelled by (2) or (4), where the respective \( \Delta \) parameter implements a normalized feedback gain, relating fictitious variables \( \eta \) and \( \zeta \). The following examples illustrates this idea.

Example 1 Consider a first order system described by the state space equation

\[
\dot{x} = ax + bu
\]

(6)

where the \( a \) coefficient is modelled by the product of two uncertain parameters \( p = p_0 + \delta_1 d_p, q = q_0 + \delta_2 d_q \) with \( \delta_i \in [-1; 1], i = 1, 2 \): \( a = pq = (p_0 + \delta_1 d_p)(q_0 + \delta_2 d_q) = p_0q_0 + \delta_1(q_0 + \delta_2 d_q) + p_0\delta_2 d_q \)

Setting the fictitious variables \( \gamma_2 = d_q x \) and \( \gamma_1 = (q_0 + \delta_2 d_q)x \), the system (6) can be modelled by

\[
\begin{pmatrix}
\dot{x} \\
\gamma
\end{pmatrix} =
\begin{bmatrix}
p_0q_0 & [d_p \ p_0] \\
[0 \ 1] & [0 \ 0]
\end{bmatrix}
\begin{pmatrix}
\eta \\
\zeta
\end{pmatrix} + bu
\]

(7)

under the closed-loop \( \eta = \Delta \gamma \) with \( \gamma = [\gamma_1 \ \gamma_2]' \) and \( \Delta = \text{diag}(\delta_1, \delta_2) \). It is remarked that the choice for (7) is not unique.

Example 1 also shows that a direct term can appear from \( \eta \) to \( \gamma \) in the generalized model. This will always be the case when a product or a ratio of two or more uncertain parameters appears in a matrix coefficient, which is often the case when modelling from physical laws. Such uncertainties will be called “higher order uncertainties”. If a coefficient is just a linear function of the uncertain parameters, then it will be called “simple” or “first order uncertainty”. It mainly occurs when one simply states that the state space matrix coefficients belong to a certain range and can be viewed as a kind of “abstract” structured uncertainty. Simple uncertainties do not contribute to the existence of a direct term from \( \eta \) to \( \gamma \).
Example 2 Consider the system of example 1 but with \( q = p \), which gives \( a = p^2 \) and \( \delta_2 = \delta_1 \). In this case (6) can be modelled by the system
\[
\begin{aligned}
\dot{x} &= p_0 x + [d_p p_0] \eta + bu \\
\zeta &= [p_0] x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \eta 
\end{aligned}
\]  
(8)
in closed loop with \( \eta = \Delta \zeta \), \( \eta = [\eta_1 \eta_2] \), \( \zeta = [\zeta_1 \zeta_2] \) and \( \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix} = \delta_1 I_2 \). It is remarked that \( \delta_2 = \delta_1 \) does not imply that the fictitious signal are the same, that is, \( \zeta_1 \neq \zeta_2 \) and \( \eta_1 \neq \eta_2 \).

2.2 Generalized system

Following the methodology of example 1 and 2 for all uncertain parameters of a system, one gets the control setup of figure 2, which is an evolution of figure 1. The vectors \( \eta \in \mathbb{R}^q \) and \( \zeta \in \mathbb{R}^n \) represent the fictitious signals associated to the uncertainties in the system equations and
\[
\begin{aligned}
\Delta &= \begin{bmatrix} \delta_1 & \delta_2 & \ldots & \delta_m \end{bmatrix} \\
\kappa &= \begin{bmatrix} k_1 & k_2 & \ldots & k_m \end{bmatrix}
\end{aligned}
\]  
(9)

relating the fictitious signals \( \eta \) and \( \zeta \). The diagonal elements of \( \Delta \) are the respective normalized gains \( \delta_i \) of each uncertain parameter, that were pulled out from the generalized model. The diagonal elements of \( \Delta \) are not necessarily independent, since the same parameter can have several instances in the system equations, leading to the same gain but applied to different signals, as shown in example 2. Defining the vectors \( \delta \) and \( k \),
\[
\begin{aligned}
\delta &= [\delta_1 \delta_2 \ldots \delta_m]^T \\
k &= [k_1 k_2 \ldots k_m]^T
\end{aligned}
\]  
(10)

the general structure of \( \Delta \) can be represented by,
\[
\Delta = \text{diag}(\delta_1 I_{k_1}, \delta_2 I_{k_2}, \ldots, \delta_m I_{k_m})
\]  
(12)

and the set \( \mathcal{D} \) of allowed values of \( \Delta \) defined by,
\[
\mathcal{D} = \{ \text{diag}(\delta_1 I_{k_1}, \delta_2 I_{k_2}, \ldots, \delta_m I_{k_m}) : \delta_i \in [-1; 1] \}
\]  
(13)

which means that exists a total of \( m \) uncertain parameters, each one “processing” \( k_i \) different signal in the system equations and \( n_\eta = n_\zeta = \sum_{i=1}^m k_i \). In example 1, where \( a = pq \), then \( m = 2 \) and \( k_1 = k_2 = 1 \). For example 2, where \( a = p^2 \), then \( m = 1 \) and \( k_1 = 2 \). See (Maciejowski, 1989) for this type of modelling.

Let \( \kappa \) be a vector that collects all the uncertain parameters for the system (1). Let \( \kappa_0 \) be the vector that collects all the nominal values and \( \kappa \) the vector that collects all the normalization constants as the \( \kappa \) elements are modelled as absolute and/or relative tolerances as in (2) or (4). For the system model (1), the coefficients of the matrices are functions of \( \kappa \),
\[
A = A(\kappa_0), \quad B_1 = B_1(\kappa_0), \quad B_2 = B_2(\kappa_0) \\
C_1 = C_1(\kappa_0), \quad C_2 = C_2(\kappa_0), \quad D_{12} = D_{12}(\kappa_0) \\
D_{22} = D_{22}(\kappa_0)
\]  
(14)

In order to be able to consider uncertainties in any coefficient of any matrix in (14) of the model (1), the generalized system \( G \) of figure 2 should be represented by the following state space equations,
\[
G \iff \begin{cases}
\dot{x} &= Ax + B_1 \eta + B_2 w + B_3 u \\
\dot{\zeta} &= C_0 x + D_{00} \eta + D_{01} w + D_{02} u \\
z &= C_1 x + D_{10} \eta + D_{12} u \\
y &= C_2 x + D_{20} \eta + D_{22} u \\
A &= A(\kappa_0), \quad B_1 = B_1(\kappa_0) \\
B_2 &= B_2(\kappa_0), \quad C_1 = C_1(\kappa_0) \\
C_2 &= C_2(\kappa_0), \quad D_{12} = D_{12}(\kappa_0) \\
D_{22} &= D_{22}(\kappa_0) \\
B_0 &= B_0(\kappa_0, \kappa), \quad C_0 = C_0(\kappa_0, \kappa, \kappa_0) \\
D_{00} &= D_{00}(\kappa_0, \kappa, \kappa_0), \quad D_{01} = D_{01}(\kappa_0, \kappa, \kappa_0) \\
D_{02} &= D_{02}(\kappa_0, \kappa, \kappa_0), \quad D_{10} = D_{10}(\kappa_0, \kappa, \kappa_0) \\
D_{20} &= D_{20}(\kappa_0, \kappa, \kappa_0)
\end{cases}
\]  
(15)
The matrices of this state space model can be split in two sets. The one that represents the nominal system \( G_{\text{nom}} \) = \{ \( A, B_1, B_2, C_1, C_2, D_{12}, D_{22} \) \} and the one that represents the uncertainties structure, \( G_{\text{unc}} \) = \{ \( B_0, C_0, D_{00}, D_{01}, D_{02}, D_{10}, D_{20} \) \}. Clearly \( G_{\text{nom}} = G_{\text{nom}}(\kappa_0) \), while for \( G_{\text{unc}} \) one gets \( G_{\text{unc}} = G_{\text{unc}}(\kappa_0, \kappa, \kappa_0) \), as shown in example 1.

The matrices \( B_0, D_{10}, D_{20} \) exists because of uncertainties in the matrices \( C_0, D_{01}, D_{02} \) are due respectively to the uncertainties in the matrices associated to \( x, w, u \) (“columns” of state space model (1)). Taking as an example the matrix \( B_1 \) and assuming \( D_{00} = 0 \), for the system (15) in closed loop with \( \eta = \Delta \zeta \), one gets a resulting
matrix of type $B_1 + B_0 \Delta D_{01}$, where $B_0 \Delta D_{01}$ models the uncertainty associated with $B_1$. More precisely, for $B_1$ the relation between (1) and (15) in closed loop with $\eta = \Delta \zeta$ is given by,

$$B_1(\kappa) = B_1(\kappa_0) + B_0(\kappa_0, d_0, p_0) \Delta D_{01}(\kappa_0, d_0, p_0)$$

(16)

It is easy to verify that (16) allows to model the coefficients of $B_1$ as belonging to intervals. Consider now that $D_{00} \neq 0$, which can be nonzero, as shown by example 1. Then (16) changes to,

$$B_1(\kappa) = B_1(\kappa_0) + \Delta D_{00}(\kappa_0, d_0, p_0)$$

(17)

and now it is not clear what type of modelling (17) represents.

2.3 Modelling Capacity

In order to keep dynamic systems linear, one wants that for the system represented by equations (1), after pulling out of the model the uncertainties in $\Delta$, one gets the linear system given by equations (15). This leads to the following question:

What functional relations can take place between the matrix coefficients of the system (1) and the uncertain parameter vector $\kappa$, such that the result is the linear system (15)?

The answer to this question is that the matrix coefficients of the system (1) can be rational functions on the elements of the vector $\kappa$. In other words, the system (15) has the "capacity" to model the matrix coefficients of (1) from simple intervals up to rational functions on the uncertain parameters. Non linear functions such as trigonometrics, exponentials, modulus, etc, just can’t be modelled by a system such as (15). An analogy can be made with the rational transfer functions on the Laplace variable $s$, that can be represented by linear block diagrams. This is best viewed with the following example.

Example 3 Consider a scalar gain $d$ modelled as a rational function on the elements of a vector $\kappa$ of uncertain parameters,

$$d = d(\kappa) = \frac{N(\kappa)}{D(\kappa)}$$

(18)

where $N(\kappa), D(\kappa)$ are polynomial functions on the elements of $\kappa$ and/or on their respective inverses.

Making use of negative feedback, the expression (18) can be rewritten as,

$$d(\kappa) = N(\kappa) \cdot \frac{1}{1 + [D(\kappa) - 1]}$$

(19)

reducing the problem to the implementation of the gains modelled by the polynomial functions $N(\kappa)$ and $D(\kappa) - 1$. Taking the elements of $\kappa$ and their respective inverses as uncertain scalar gains, these can be implemented using the four basic linear blocks shown in figure 3 and 4. Each generic scalar gain $k$ or its inverse $1/k$ is modelled as a static system with two inputs/two outputs and generates a pair of scalar signals $\{\eta, \zeta\}$. In closed loop with $\eta = \Delta \zeta$, one gets implementations for $v_o = kv_i$ or $v_o = (1/k)v_i$. The gains $N(\kappa)$ and $D(\kappa)$ - 1 can be built by linear block diagrams from the basic blocks. One can conclude that $d(\kappa)$,

![Diagram](https://via.placeholder.com/150)

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**Fig. 3. Basic linear blocks for the implementation of direct scalar gains $v_o = kv_i$, where $k$ is a parameter subject to uncertainty.**
implies the existence of a direct term between confirmed by the basic blocks of figure 3 and 4. uncertain parameter in a matrix coefficient of (1), shows that the existence of a product of uncertain parameters or the existence of an inverse of an vector that collects the respective signals from in closed loop with a static linear system seen as a gain \( y = d(k) u \), can itself be modelled by a static linear system

\[
\begin{align*}
\zeta &= D_{00}\eta + D_{01}u \\
y &= D_{10}\eta + D_{11}u
\end{align*}
\]

in closed loop with \( \eta = \Delta \zeta \), where \( \{\eta, \zeta\} \) are the vectors that collects the respective signals from each basic block.

System modelling trough block diagrams also shows that the existence of a product of uncertain parameters or the existence of an inverse of an uncertain parameter in a matrix coefficient of (1), implies the existence of a direct term between \( \eta \) and \( \zeta \), that is, \( D_{00} \neq 0 \) in (15). This fact is easily confirmed by the basic blocks of figure 3 and 4.

The fact that the system (15) is capable of modelling the matrix coefficients of (1) as rational functions of the uncertain parameters allows for a very significative modelling flexibility. Most often, rational functions are enough. In the presence of non linear functions, these can be replaced by rational or polynomial approximations in the range of interest of the parameters. It is also possible to interpolate systems defined from operation points.

**Example 4** Consider the system of example 1 with the \( a \) coefficient given by,

\[
a = \cos(p) = \cos(p_0 + \delta_1 d_p) \quad (21)
\]

In this case one gets a non linear system in the uncertainty. Suppose that a second order polynomial approximation \( \beta(p) \simeq \cos(p) \) is enough for the desired range of values of \( p \). Then,

\[
a = \beta_0 + \beta_1(p_0 + \delta_1 d_p) + \beta_2(p_0 + \delta_1 d_p)^2 = \beta_0 + \beta_1 p_0 + \beta_2 p_0^2 + d_p \beta_2(2\beta_1 p_0 + \delta_1 d_p) + \beta_1 \delta_1 d_p \quad (22)
\]

which is a similar case of example 1 with \( \delta_2 = \delta_1 \).

**Example 5** Consider the case where the model structure (6) is used for modelling a non linear system in three operating points, corresponding to the values \( a \in \{a_1, a_2, a_3\} \), with \( a_1 < a_2 < a_3 \). Suppose that this three points are interpolated by a second order polynomial \( \beta(\delta_1) \) such that \( \beta(-1) = a_1 \) and \( \beta(1) = a_3 \),

\[
a = \beta(\delta_1) = \beta_0 + \beta_1 \delta_1 + \beta_2 \delta_1^2 = \beta_0 + \delta_1(\beta_1 + \beta_2 \delta_1) \quad (23)
\]

which gives once more a similar case of example 1.

**Example 6** Consider the system of example 1 with \( p \in [1; 2] \), \( q \in [4; 6] \) and \( b = 1 \). Modelling \( p \) and \( q \) with absolute tolerances gives,

\[
p_0 = 1.5, \quad d_p = 0.5, \quad q_0 = 5.0, \quad d_q = 1.0 \quad (24)
\]

Using the basic blocks of figure 3 and 4, the system (15) can be represented by the Simulink diagram of figure 5. Computing a linearization,

one gets the numeric values for a realization of the corresponding system (15)

![Simulink diagram for example 6](image.png)
\[
\begin{align*}
\dot{x} &= 7.5x + [2.5 \ 1.0]y + 1.0u \\
\zeta &= \begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix} y
\end{align*}
\] (25)

This realization is different from the one obtained analytically in (7), which confirms that several realizations are possible.

3. CONCLUSIONS

This paper approaches the problem of system parametric modelling. It is assumed that the system to model is a linear time invariant dynamic system described by a set of equations with real parameters whose values are unknown but that belongs to a known interval. This is most often the case of systems described by physical laws.

A systematic method for modelling this type of systems is developed and analyzed. The method treats uncertain parameters as gains and relies on block diagrams based on four basic blocks implementing direct and inverse uncertain gains. It is concluded that one gets for the modelled system a linear state space model if the original functional relation between the coefficients of the equations describing the system and the uncertain parameters are rational or polynomial functions. In the case of other types of functional relations, a local approximation by polynomials in the proper range is a viable solution.

It is remarked that rational and polynomial relations on the uncertain parameters are considered complex enough to cope with a large number of cases and that the method can be systematically applied to complex systems.

REFERENCES

