Active noise control algorithms with reduced channel count and their stability analysis

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Abstract

In narrowband active noise control systems, the dimension of the quiet zone around each error sensor is proportional to the wavelength of the noise being cancelled. In broadband active noise control systems, several error sensors per anti-noise source are used (two as a rule of thumb) in order to achieve larger cancellation regions, but the fact still remains that the size of the quiet zone will be proportional to the wavelength of the noise. In this paper, it is proposed to achieve wide quiet zones by bandlimiting the anti-noise signal. This results in techniques that do not need the extra number of error sensors. This cannot be done by placing a filter in the cancellation path, or using the anti-aliasing or reconstruction filters, unless you use a very sharp filter with long delay. Instead, several algorithms based on the MFxLMS algorithm are presented that accomplish this task. This permits systems with more anti-noise sources than error sensors, which allows a better inversion of the cancellation path while still achieving large quiet zones. The new algorithms are less sensitive to cancellation path modelling errors. Namely, the maximum error allowed in the delay estimate increases as the anti-noise bandlimiting frequency decreases.

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1. Introduction

Active noise control (ANC) system generate sound waves with opposite phase to the noise (anti-noise) that interfere destructively with it to reduce the noise level [1,2] in a given region called “quiet zone”.

A feedforward ANC system is represented in Figs. 1 and 2. In order to achieve larger quiet zones, many ANC systems are multichannel, with several reference sensors, anti-noise transducers and error sensors. The systems represented in Figs. 1 and 2 have N reference sensors, P anti-noise sources and M error sensors. The reference sensors measure the noise signals, \( x_0(n) \ldots x_N(n) \), at some points so that it can be used to predict the noise signals at the error sensors. This signals are sampled and digitally processed by a digital signal processor (DSP), which implements the ANC controller. The ANC controller produces the anti-noise signals, \( y_0(n) \ldots y_P(n) \). The reference and error anti-aliasing filters and the anti-noise reconstruction filter banks are represented, respectively, by \( A_1(f) \), \( A_2(f) \) and \( R(f) \). These correspond to a bank of identical filters that filter each input. These signals are then fed to the anti-noise transducers and travel through the physical medium, usually air, to the error sensors.
The error sensors measure the error signals, $e_0(n) \ldots e_M(n)$, which are used to adapt parameters of the ANC controller. The paths travelled by the anti-noise signals, from the transducers to the error sensors, including the reconstruction and anti-aliasing filters, are the cancellation or secondary paths, and $\hat{S}(f) = R(f)S_A(f)A_2(f)$ is the cancellation paths matrix, or simply cancellation path. Note that $S(f)$ represents each path, from each source to each error sensor. We will use this notation through the text. We call its acoustic components, $S_A(f)$ the acoustic cancellation path. The paths travelled by the noise signals from the reference sensors to the error sensors are the primary paths, and $P(f) = A^{-1}_1(f)P_A(f)A_2(f)$ is the primary paths matrix, or simply primary path, where $P_A(f)$ is the acoustic primary path. The pre-reference paths and the feedback paths are not shown in the figure. The pre-reference paths are the paths from the noise sources to the error sensors. This can be ignored when designing the ANC DSP algorithms, by incorporating its inverse into the primary path.

Note, however, that if the system setup is poorly designed this inverse may be difficult to implement and performance can be affected [3]. The feedback paths are the paths from the cancellation loudspeakers to the reference sensors. This can be removed using an echo canceller [1].

The best known ANC algorithm is the Filtered X Least Mean Squares algorithm (FxLMS) [4, 5] as represented in Fig. 1. In this algorithm the reference signal is filtered by an estimate of the cancellation path $\hat{S}(f)$ and then used as an input to the LMS algorithm. This estimate must be determined offline, prior to the start of the FxLMS, or be constantly updated on-line using some algorithms such as the Overall Modelling Algorithm (OMA) or the additive with noise [1]. The FxLMS algorithm can be modified to form the modified FxLMS (MFxLMS) [6–9] as shown in Fig. 2. In the MFxLMS algorithm the anti-noise signal is subtracted from the error signal $e_k[n]$, resulting in $d_k[n]$, which is then used as a desired signal to the LMS algorithm. Thus, the effect of the cancellation path is actually removed, as along as its estimate, $\hat{S}(f)$, is accurate. This allows the use of more sophisticated adaptive filtering algorithms such as the RLS or the Kalman filter [8]. It will also allow the implementation of the anti-noise bandlimiting action as introduced in this paper.

The dimension of the quiet zone is related to the wavelength of the noise under consideration. For example, in narrowband single-channel ANC systems the quiet zone has a radius of about one-tenth of the wavelength, $\lambda/10$. For broadband ANC system, the quiet zone dimensions will be limited by the highest frequency component in the anti-noise signal. Usually, in broadband systems, large quiet zones are accomplished by the use of several error sensors per anti-noise source, two as a rule of thumb, so that if one anti-noise source is able to achieve noise reduction in two sensors then it should be creating a fairly large quiet zone. In this paper large quiet zones are achieved by bandlimiting the anti-noise signal. Some techniques are proposed to do this. This will allow systems which have a greater number of anti-noise sources than error sensors which allows a better inversion of the cancellation path [3] permitting the achievement of higher levels of noise reduction.

2. The multi-channel FX and MFX algorithms

Given the multichannel system proposed with $N$ references, $P$ anti-noise sources and $M$ error sensors
each signal in a set is going to be labelled with an index. Let \( i \) represent the reference number index, \( j \) the anti-noise source number index, and \( k \) the error sensor number index. Define the following signals: \( x_{i,j}[n] = x_i[n - l] \), where \( x_i[n] \) is the \( i \)th reference signal; \( S_{i,j,k} = S_{j,k}(l) \), where \( S_{j,k}(l) \) is the impulse response of the cancellation path from the \( j \)th anti-noise source and the \( k \)th error sensor; \( W_{i,j,l} = W_{i,j,l}(l) \), where \( W_{i,j,l}(l) \) is the impulse response of the controller filter that relates the \( i \)th reference and the \( j \)th anti-noise source. Assume that the controller impulse response is of length \( L \), and cancellation path impulse response is of length \( R \). Given this, the multichannel FxLMS (normalized) algorithm can be described as

\[
x'_{i,j,k}[n] = \sum_l x_{i,l}[n]S_{i,j,k}, \quad x'_{i,j,k}[n] = x'_{i,j,k}[n - l], \quad (1)
\]

\[
\mu = \alpha \left( \sum_{i,j,k} x_{i,j,k}^2[n] \right)^{-1}, \quad (2)
\]

\[
w_{i,j,l}[n] = \lambda w_{i,j,l}[n - 1] + \mu \sum_k x'_{i,j,k}[n]e_k[n], \quad (3)
\]

\[
y_j[n] = -\sum_{i,l} x_{i,l}[n]w_{i,j,l}[n] \quad (4)
\]

and the modified FxLMS by replacing (3) by

\[
d_k[n] = e_k[n] - \sum_{i,l} y_{i,l}[n]S_{i,j,k}, \quad (5)
\]

\[
x_k[n] = d_k[n] - \sum_{i,l} x'_{i,j,k}[n]w_{i,j,l}[n - 1], \quad (6)
\]

\[
w_{i,j,l}[n] = \lambda w_{i,j,l}[n - 1] + \mu \sum_k x'_{i,j,k}[n]x_k[n]. \quad (7)
\]

The step size of the algorithm is given by \( \mu \) and is normalized to the reference signal power, which assures the stability of the algorithm when there are no cancellation path errors, as shown below. The leakage factor of the algorithm is given by \( \lambda \). This is similar to adding white noise to the input of the algorithm, implementing effort weighting, and resulting in the improvement of the stability of the algorithm in situations where there is not sufficient excitation provided by the reference signal [1].

All the algorithms presented in this paper are based on the modified FxLMS algorithm, with a few additional modifications, and can thus be implemented using the equations presented above.

### 3. Choosing the step size

Consider the case of the modified FxLMS algorithm which will be used through the paper. If there are no secondary path modelling errors, the effect of the secondary path is perfectly cancelled. By substituting (6) into (7) the algorithm can be described by

\[
w_{i,j,l}[n] = \lambda w_{i,j,l}[n - 1] + \mu \sum_k x'_{i,j,k}[n]d_k[n]
\]

\[
- \mu \sum_k \left( x'_{i,j,k}[n] \sum_{l,j} (x'_{i,j,k}[n] w_{i,j,l}[n - 1]) \right).
\]

(8)

In order to achieve some analytical results, one can join the indices of \( w_{i,j,l} \), into a single index, for example, given by \( k = l + i \times L + j \times L \times N \). Let the resulting vector be represented by \( w_{(i,j,l)} \), where the \( (...) \) represents joined indices. Applying the same procedure to the \( x'_{i,j,k} \) one gets the matrix \( x_{(i,j,k)} \) with \( V = L \times N \times P \) rows and \( M \) columns. Now (8) can be written in a matrix form as

\[
\textbf{W}[n] = \lambda \textbf{W}[n - 1] + \mu \textbf{P}[n] - \mu \textbf{R}[n] \textbf{W}[n - 1]
\]

(9)

with

\[
\textbf{W}^{V \times 1}[n] = w_{(i,j,l)}[n],
\]

(10)

\[
\textbf{P}^{V \times 1}[n] = x'_{(i,j,k)}[n]d_k[n],
\]

(11)

\[
\textbf{R}^{V \times V}[n] = x'_{(i,j,k)}[n]x'_{(i,j,k)}[n].
\]

(12)

The matrix \( \textbf{R}[n] \) has characteristic number \( V \) and positive eigenvalues, since it is Hermitian. By choosing a transform that diagonalizes the matrix \( \textbf{R}[n] \) the equation becomes a series of \( M \) decoupled first-order recursive filter equations

\[
w'_m[n] = (\lambda - \mu r'_m[n])w'_m[n - 1] + \mu p'_m[n],
\]

(13)

where \( r'_m[n] \) are the eigenvalues of \( \textbf{R}^{V \times V}[n] \). This equation will be stable as long as \( \mu < (\lambda + 1)/r'_m[n] \). Note that this is only for perfect cancellation path modelling because with cancellation path modelling errors \( p'_m[n] \) depends on \( w'_m[n - 1] \). Since the eigenvalues are positive then they are all smaller than their sum, which is the trace of the matrix. This can also be calculated by summing over all the diagonal elements of the matrix, \( \textbf{R}[n] \), as in

\[
\sum_{i,j,k} x'^2_{i,j,k}[n] > r'_m[n] \quad \text{for all} \quad m.
\]

(14)
So choosing the step size as in (2) with \( z < \lambda + 1 \approx 2 \) assures the stability of the algorithm in the absence of secondary path errors. Since for multichannel system this bound is pessimist setting alpha to one is a reasonable choice.

4. Limiting the anti-noise frequency

When trying to limit the anti-noise signal band, the first idea that comes to mind is to place a low pass filter at the input of the anti-noise source, or reducing the sampling frequency.

If we simply reduce the sampling frequency, then the cut band of the reconstruction and anti-aliasing filters have a high stop-band attenuation and a high delay. To keep this delay to a minimum it is desirable that the sampling frequency be high. More, usually these filters have a high stop-band attenuation and a high delay, more than is strictly necessary for the bandlimiting action.

If we place a low pass filter at the input of the anti-noise source, then, the controller will then try to invert this filter. The use of a sharp transition band and a leakage factor or adding whole noise, \( z[n] \), at the input of LMS algorithm, as represented in Fig. 2 can prevent this, as shown ahead. For the single channel case, ignoring causality constrains all the calculations can be done in the frequency domain, independently for each frequency. Then the optimal controller as given by the Wiener filter [1] is

\[
W(f) = \frac{S_{df}(f)}{S_{x,\hat{y}}(f)} = \frac{P(f)S_{xx}(f)S^{*}(f)}{|S(f)|^{2}S_{xx}(f) + S_{zz}(f)},
\]

where \( S_{ab}(f) \) is the cross-power spectral density between the signal \( a[n] \) and \( b[n] \), that is, the discrete time Fourier transform of the cross-correlation function, \( r(l) = E[a[n]b[l - n]] \). Note that if \( a[n] \) is equal to \( x[n] \) filtered by a filter with transfer function \( A(f) \) and \( b[n] \) is \( x[n] \) filtered by a filter with transfer function \( B(f) \) where \( x[n] \) is a stochastic signal with autocorrelation \( S_{xx}(f) \) then \( S_{ab}(f) = A(f)S_{xx}(f)B^{*}(f) \).

This gives

\[
W(f) = \begin{cases} 
P(f)/S(f), & S \text{ pass-band} \\
0/S_{zz}(f), & S \text{ stop-band}, 
\end{cases}
\]

which is the desired answer. Note that \( S_{zz}(f) \) prevents a 0/0 indetermination. In the transition band the controller can assume rather high values, which is undesirable. Sharp filters, with narrow transition bands, have long delays, which is undesirable in ANC systems. It is better to use some of the structures presented in the paper, as discussed below.

One technique is to use the modified filtered-X configuration to change the desired signals (which correspond to the desired anti-noise signals) to a band-limited version of the originals. This is presented in Fig. 3. In this configuration the bandlimiting action is implemented by the controller. It will adapt into a bandlimiting filter that removes the high frequency components of the anti-noise signal, i.e., the controller will adapt to minimize a signal with power spectral density given by

\[
S_{xx}(f)P(f) - e^{-\omega DF} \hat{S}(f)W(f)^{2}.
\]

The optimal filter, disregarding causality constrains, is then given by

\[
W(f) = \frac{P_{n}(f)}{\hat{S}(f)F(f)e^{\omega DF}}
\]

with \( \hat{S}(f) = S(f) = R(f)S_{xx}(f)A_{2}(f) \) and \( A_{1}(f) = A_{2}(f) \). This is the band limited version of the optimal controller as desired. Note that \( DF \) is the group delay of the filter \( F \), and so \( F(f)e^{\omega DF} \) is a filter with no group delay.

It is possible to remove this extra burden from the controller, reducing the order required for it, by using a separate filter to implement the bandlimiting action. This is represented in Fig. 4. This in fact corresponds to modifying the secondary path by
adding an extra bandlimiting filter. This has the disadvantage that the bandlimiting filter is fixed, i.e. it has a predefined attenuation and delay. In the configuration of Fig. 3, it is possible to design the filter, $F(f)$, to have a large attenuation, and the controller will adapt to the actual physical plant producing, a higher delay and attenuation filter when possible and a lower attenuation filter when a high-delay filter is not tolerable. A combination of the two techniques is also possible.

Splitting the $F(f)$ filter into $A_1(f)R(f)$ and rearranging, i.e., exchanging $A_1(f)$ with $W(f)$ and lowering the sample rate (note that $S(f)$ includes $R(f)$) results in the configuration of Fig. 5. This configuration allows complexity reduction due to lower sampling rate; however, now $A_1(f)$ and $R(f)$ are anti-aliasing and reconstruction filters and must have a high attenuation and greater delay. Note that this configuration is equivalent to the original MFX but with compensation of the anti-aliasing and reconstruction filters transfer functions. This allows the use of less sharp and lower delay anti-aliasing and reconstruction filters.

All of the configurations presented allow the design of systems with more anti-noise sources than error sensors while still achieving large quiet zones. In these systems, the inversion of the cancellation path is easier [3], allowing greater noise reductions. Configuration I is the most flexible and that allows the best performance.

5. Effect of cancellation path estimation errors

In order to simplify the results, a one-channel narrowband analysis will be presented. The analysis is similar to the one in [9], but extended to the new algorithms presented in the paper.

In frequency-domain analysis [4,10] only the amplitude and phase response of the system is taken into account. However, in ANC systems the delay in the cancellation path can have a large influence on the behaviour of the systems. In narrowband analysis, the group delay is also taken into account. The signal is taken to be a narrowband amplitude-modulated signal with a given carrier frequency, so the system will alter the carrier phase and delay the modulated signal [11]. More exactly, the secondary path will be defined by amplitude $S$, phase $\theta_S$ and delay $d$, which correspond to the complex amplitude $S_z$, and a group delay $d$, at a given frequency. The bandlimiting filter will also be defined by the amplitude, phase, and group delay, $F, \theta_F, D_F$. Narrowband analysis results in much simplification, but it still maintains information about the dynamics of the system to provide some useful insight. Frequency domain or narrowband analysis of a time-domain implementation is a good approximation for long adaptive filters where the modes of convergence are approximately sinusoidal and mode overlap is low. The equations will be first derived for a sinusoidal input, as follows.

Define the modulated signals, filtered reference signal, $x'[n]$, and innovation signal, $z[n]$, in terms of the baseband signals $x'_z[n] = x'_z[n]$ and $z_z[n] = z_z[n]e^{i\phi_z[n]}$ centred at carrier frequency $\omega_0$:

$$x'[n] = x'_z[n] \cos(\omega_0 n + \phi_z),$$  \hspace{1cm} (19)

$$z[n] = z_z[n] \cos(\omega_0 n + \phi_z[n])$$  \hspace{1cm} (20)
for all proposed algorithms the \( L \)-tap controller filter, \( \omega_k[n] \), for \( k \) from zero to \( L - 1 \), is updated as
\[
w_k[n] = w_k[n - 1] - \mu \nu [n - k] z[n],
\]
where \( \mu \) is the step size of the algorithm. Now we have
\[
x'[n - k] z[n] = \frac{1}{2} x'_A[n - k] z_A[n] (\cos(\phi_x - \omega_0 k - \phi_x[n]) + \cos(2\omega_0 n + \phi_x[n] - \omega_0 k + \phi_x))
\]
(22)

the last cosine terms can be neglected for small step sizes, since it is of high frequency and should be filtered by the update equation of the filter. This approximation is worse if the signal has a very low normalized frequency \( \omega \) but this kind of signal also require long filters and smaller step sizes, resulting in better accuracy. Since the input signal is around frequency \( \omega_0 \), then only the controller response at this frequency is of interest. Calculating the discrete-time Fourier transform of the controller at \( z = e^{j\omega_0} \) results in
\[
\hat{w}_z[n] = \sum_{k=0}^{L-1} w_k[n] e^{-j\omega_0 k}
\]
(23)

and we can write the time update equation as
\[
\hat{w}_z[n] = \hat{w}_z[n - 1] - \mu \frac{z_A[n]}{2} \sum_{k=0}^{L-1} x'_A[n - k] h[k]
\]
(24)

with
\[
h[k] = \cos(\phi_x - \omega_0 k - \phi_x[n]) e^{-j\omega_0 k}.
\]
(25)

The summation represents a convolution operation. The Fourier transform of \( h[n] \), \( H(e^{j\phi}) \) has a peak at \( \phi = 0 \) and \( 2\omega_0 \). Since \( x'_A[n] \) is a narrow-band baseband signal, we are mainly interested in the amplitude, phase and group delay at \( \omega = 0 \), namely
\[
H(e^{j\phi}) = \frac{L_x}{2} e^{-j\phi[n]}
\]
(26)

with
\[
L_x = e^{j2\phi[n]} e^{-j\omega_0 L} \sin(\omega_0 L) \sin(\omega_0) + L
\]
(27)

for \( \omega_0 = k\pi/L \), \( L = L_x \). For large \( L \), \( L_x \) is approximately real and \( L_x \approx L \). The group delay of \( H \) at \( \omega = 0 \) is given by \( d_L = (L - 1)/2 \). Finally, this results in
\[
\hat{w}_z[n] = \hat{w}_z[n - 1] - \mu \frac{L_x}{2} x'_A[n - d_L] z_A[n]
\]
(28)

with
\[
x'_A[n] = x'_A[n] e^{j\phi_x},
\]
(29)

\[
z_A[n] = z_A[n] e^{j\phi_x[n]}
\]
(30)

Note that the product \( x'_A[n - d_L] z_A[n] \) corresponds to a correlation operation. The procedure that follows is similar to the one in [9] with some modifications, namely now taking into account the effect of \( L_x \), \( d_L \), and most importantly of the bandlimiting filters. For Configuration I, in Fig. 3, we can write
\[
\hat{w}_z[n] = \hat{w}_z[n - 1] - \mu \frac{x'_A[n - d_L - \hat{d} - D_F] \hat{S}^*_z z_A[n]}{2}
\]
(31)

where \( \mu_x = \mu L_x/2 \). The secondary path model is defined by placing a hat over the corresponding symbols for the secondary path, namely, \( \hat{S} \), \( \theta_S \), \( \hat{d} \), \( \hat{S}_z \). The innovation term is given by
\[
z_A[n] = x_z[n - \hat{d} - D_F - d_w] z_A S_z F_z
\]
\[
+ x_z[n - \hat{d} - D_F - d_w] \hat{w}_z \hat{S}_z F_z
\]
\[
- x_z[n - d_L - d] \hat{S}_z F_z
\]
\[
+ x_z[n - \hat{d} - D_F - d_w] \hat{S}_z \hat{w}_z
\]
\[
\times [n - 1] + r[n],
\]
(32)

where \( d_w \) is the delay of the controller filter and \( r[n] \) is a measuring noise term, which is uncorrelated with the reference signal.

In this paper, only the convergence of the mean is going to be studied. Replacing \( z[n] \) in (31), taking expected values, and letting \( R_{\hat{d}d} = E[x'_z[n - d_L - \hat{d} - D_F - d_w] x_z[n - \hat{d} - D_F]] \), \( R_{\hat{d}d} = E[x'_z[n - d_L - d - D_F - d_w] x_z[n - d - D_F]] \) and taking the Z-transform [12], one obtains
\[
E\{\hat{w}_z(Z)\} = \frac{\mu_z F_z R_{\hat{d}d} \hat{S}^*_z S_z Z}{Z - 1 + \mu_z |\hat{S}_z|^2 R_X} w_0(Z)
\]
(33)

with
\[
R_X = R_{\hat{d}d}(1 - F_z Z^{-\hat{d} - D_F}) + F_z R_{\hat{d}d} S_z / \hat{S}_z Z^{1 - \hat{d} - D_F}.
\]
(34)

Using this equation, one can express \( \mu_x \) as a function of \( Z \), \( \mu_x = \Gamma(Z) \), where \( Z \) is a pole of the system. This means that if a pole of the adaptive filter convergence is known, then it is possible to calculate the step size used by the algorithm. However, in general this function has no inverse,
since there are several modes of convergence, or poles, for a given step size. Nonetheless, for the case of very small step sizes, the convergence is dominated by a single pole, near \( Z = 1 \), and the inverse exists. Using the rule for the inverse of the implicit function, that is,

\[
F(\mu_x, Z) = 0 \quad \Rightarrow \quad \frac{\partial Z}{\partial \mu_x} = -\frac{F^1(\mu_x, Z)}{F^2(\mu_x, Z)}
\]  

(35)

one can obtain a linear approximation for very small step sizes. In this equation, \( F^1(\mu_x, Z) \) represents the derivative in terms of the first argument and \( F^2(\mu_x, Z) \) is the derivative in terms of the second argument of the function. Since, once more, the algorithm is dominated by the pole at \( Z = 1 \), calculating the partial derivative for \( Z = 1 \) and \( \mu_x = 0 \) one gets

\[
Z(\mu_x) \approx Z(0) - \mu_x \frac{\partial Z}{\partial \mu_x}
\]

\[
= 1 - \mu_x[(1 - F_z)R_{dd}|\hat{S}_z|^2 + F_zR_{dd}\hat{S}_zS_z^*].
\]

(36)

The equation should be analysed for the case when the signals are inside the noise reduction band, with \( F_z = 1 \) and outside of the noise reduction band \( F_z = 0 \). If the reference signal is not narrowband then it can be split into two signals inside and outside of the noise reduction band. We will assume that \( \mu_x \) is real and greater than zero. This is not actually true because \( L_x \) is not always real. The effect of complex valued \( L_x \) can be interpreted as reducing the tolerance of the algorithms to phase errors in the cancellation path estimate, in a worst-case scenario, although in practice it can improve or degrade the stability in a rather aleatory way. When the reference signal is inside the noise reduction band, \( F_z = 0 \), the system is always stable since \( R_{dd}|\hat{S}_z|^2 \) is always real and positive. Actually for this signal there is no feedback from the physical system. When the reference signal is inside the noise reduction band, then for the system to be stable \( R_{dd}\hat{S}_zS_z^* \) must have a positive real component, otherwise the poles will step out of the unit circle. For a narrowband reference system, the values of \( x_c[n] \) can be taken to be approximately constant, so \( R_{dd} \) is approximately equal to \( R_{dd} \) and positive real. This implies that the condition for stability is simply that the phase error \( \Delta \theta_S = \theta_S - \theta_k \) must be smaller than \( 90^\circ \), \( |\Delta \theta_S| < 90^\circ \). This is the same result as the one obtained for the FxLMS algorithm and is in agreement with what would be intuitively expected.

Now it remains to determine the maximum values for the step size, which assures the stability of the algorithm. That is, the value of the stepsize that results in the first crossing of the unit circle by a pole. Once again assuming the step size is a real number, a point \( Z \) is a pole of the system if \( \mu_x = \Gamma(Z) \), as given by (33), is a real number. If the pole is in the unit circle, then, \( Z = e^{i\theta_Z} \). The paper now proceeds to determine the values of \( \theta_Z \) for the poles, and then calculate \( \mu_x = |\Gamma(e^{i\theta_Z})| \) to obtain the limiting values of the stepsize. To obtain analytical expressions, some simplifications are required, which result in sufficient conditions for stability, but which are not always necessary. It is possible to write

\[
\mu_x = \Gamma(e^{i\theta_Z}) = -\frac{2j \sin(\theta_Z/2)e^{i\theta_Z/2}}{R_{dd}|\hat{S}_z|^2(1 - F_z\delta_S e^{-j(\Delta + D_F)\theta_Z})}
\]

(37)

with

\[
\delta_S = 1 - \frac{R_{dd}\hat{S}_z}{R_{dd}\hat{S}_z} e^{-\Delta d} \quad \text{and} \quad \Delta d = d - \hat{d}.
\]

(38)

Once again for \( F_z = 0 \) the system is always stable. Assuming \( F_z = 1 \), a lower bound for the absolute value of \( \mu_x \) is

\[
\frac{2 \sin(\theta_Z/2)}{R_{dd}|\hat{S}_z|^2(1 + |\delta_S|)} \leq |\mu_x|.
\]

(39)

To determine the values of \( \theta_Z \) of the poles, one must make the imaginary part of \( \mu_x \) equal to zero, as in (40). The smallest values for \( \theta_Z \) is the one which results in a lower limit for the stepsizes:

\[
\text{Im} \{\mu_x\} = 0 \quad \Leftrightarrow \quad \text{Re} \{(1 - \delta_S e^{-j(\Delta + D_F)\theta_Z})e^{-i\theta_Z/2} \} = 0
\]

(40)

which is equivalent to

\[
\cos \left( \frac{\theta_Z}{2} \right) = |\delta_S| \cos(\phi), \quad \phi = \left( \hat{d} + D_F + \frac{1}{2} \right) \frac{\theta_Z}{2}.
\]

(41)

If \( |\delta_S| > 1 \) and if \( \hat{d} \) and \( d \) are large, then this equation has solutions for small values of \( \theta_Z \), which limits the step size to very small values, making the algorithm unstable in practice. Otherwise the equation only has solutions for \( \theta_Z > \theta_Z' \), with

\[
\cos(\theta_Z/2) = |\delta_S| \quad \Leftrightarrow \quad \theta_Z' = 2 \cos^{-1}(|\delta_S|)
\]

(42)

Once again, for large \( d \), small changes in \( \theta_Z \), result in large changes in \( \phi \), namely \( \cos(\phi) \) goes from \(-1\) to \(1\) in only \( 2\pi/(d + D_F + 1/2) \), so the first zero
crossing can be taken as $\theta_Z \approx \theta'_Z$. Replacing (42) in (39), results in

$$\mu_x = \frac{2}{R_{dd}|S_z|^2} \sqrt{\frac{1 - |\delta_S|}{1 + |\delta_S|}}.$$  \hspace{1cm} (43)

This equation results in a lower limit for the step size which guarantees stability. The equation only has solutions for $|\delta_S| < 1$.

To determine the greatest cancellation path delay estimation error that permits the algorithm to be stable, one must determine $\Delta d$ so that $|\delta_S| = 1$. This will be done for the case of no phase errors. Different results are obtained for a sinusoidal reference signal, and a white noise signal.

For a sinusoidal reference, the signal amplitude is constant so $R_{dd} = R_{dd}$ resulting in

$$|1 - e^{2\pi f/F_S d}|^2 = 1 \iff \Delta d_{max} = F_S f / 6,$$ \hspace{1cm} (44)

where $F_S$ is the sampling frequency, and $f$ is the frequency of the sinusoidal excitation signal. For a white noise signal only the components within the anti-noise cut frequency is relevant. In this case the greatest signal component will be random phase sinusoid with frequency $F_c$. This results in

$$R_{dd} = \cos(2\pi F_c / F_S \Delta d) R_{dd}$$ \hspace{1cm} (45)

and so $|\delta_S| = 1$ is equivalent to

$$|1 - \cos(2\pi f / f_a \Delta d)e^{2\pi f / f_a \Delta d}|^2 = 1 \iff \Delta d_{max} = f_a / f / 4.$$ \hspace{1cm} (46)

This results agree well with simulations.

6. Causality

The bandlimiting filter, $F(f)$, should be flat in the pass-band, and have a constant group delay, $D_F$, so it is easily implemented using linear phase FIR filters [12]. This can be designed using a number of methods. We will use the Parks–McClellan algorithm although it is possible to design lower delay filters.

For this application, the chosen filter must be accurate in the pass-band. Any error in the pass-band will add to the residual noise since the controller will try to follow the wrong signal. So the pass-band ripple must be smaller than the desired noise reduction level. A low attenuation at the stop-band will make regions where there will be some added noise. This added noise will be equal to the signal measured at the reference times the attenuation. As long as the added noise is considerably smaller than the residual noise, this should not be to serious. Using these considerations, one can obtain the specifications of the filter. Now, it is possible to extract the value for the delay of the filter in standard time units. Since these are type-I linear phase filters, the delay is simply given by $d = MT/2$, where $T$ is the sampling time. This results in

$$D_f = \frac{-10 \log(\delta_1 \delta_2) - 13}{29.2 \Delta f},$$ \hspace{1cm} (47)

where $\delta_1$ is the pass-band ripple, $\delta_2$ is the stop-band ripple, $M$ is the filter order and $\Delta f$ is the transition-band in Hertz. For a 200 Hz transition band, 20 dB stop-band attenuation and 0.1 transition band ripple, a delay of 1.2 ms will result. This delay is inversely proportional to the transition bandwidth, and this must be much smaller than the useful band of the system, which limits the chosen value. In the proposed algorithm, as shown in Fig. 3, the optimum controller, given by the Wiener filter, is given by

$$W(f) = \frac{S_{dx}(f)}{S_{xx}(f)} = \frac{F(f) P(f) S_{xx}(f) R_A^*(f) S_A^*(f) A_2(f) e^{2\pi i D_f}}{|R(f) S_A(f) A_2(f)|^2 S_{xx}(f) + S_{zz}(f)},$$ \hspace{1cm} (48)

where the effect of anti-aliasing and reconstruction filters is included. If $S_{xx}(f)$ is small, and $A_1(f) = A_2(f)$, this is approximately given in the pass-band by

$$W(f) = \frac{S_{dx}(f)}{S_{xx}(f)} = \frac{F(f) P_A(f) e^{2\pi i D_f}}{R(f) S_A(f) A_2(f)} = F(f) e^{-2\pi i (d - D_f)},$$ \hspace{1cm} (49)

where the primary path cancellation path and the anti-aliasing and reconstruction filters were all taken as pure delays for the useful band of the system (limited by $F$). The delay $d$ is given by $d = d_p - d_S - d_A - d_R$, where $d_p$, $d_S$, $d_A$ and $d_R$ are the, acoustic primary and acoustic secondary path, anti-aliasing filter and reconstruction filter delays. Let us assume that $F$ is designed to minimize its group delay $D_f$. This means that one cannot implement $W(f)$ if $d - D_f < 0$. Actually, in a digital signal processing system, there is usually a one-sample delay or more, since the current output cannot depend directly on the current input.
gives \( d > D_F + T_S \), where \( T_S \) is the sampling period of the system, or,
\[
d_p > d_S + d_A + d_R + D_F + T_S. \tag{50}
\]
If we chose a high sampling frequency, so that \( d_A \), \( d_R \) and \( T_S \) are small compared to \( D_F \) then this is simply \( d_p > d_S + D_F \). This is a good criterion for selecting the sampling frequency.

One must not forget that the above conditions were derived for the optimal filter disregarding the causality constrains. If this were included the controller would try to implement a predictor when required. So the system might still be able to achieve noise reduction even if these conditions are not met, but only through prediction of the noise signal.

7. Anti-aliasing and reconstruction filters

The problem of designing the anti-aliasing and reconstruction filters is a bit different from the problem of designing the anti-noise bandlimiting filter. The anti-aliasing filters can have a larger transition band, if one is willing to increase the sampling rate of the system. The transition band can go from the anti-noise bandlimiting frequency, \( F_c \), to half the sampling frequency of the system, \( F_s/2 \).

The transition-band ripple can also be higher since the controller will be able to compensate for this. The stop-band attenuation should be high, in order to prevent aliasing of extra noise into the pass-band. However, if the out-off-band signal is broadband as in sigma–delta converters, a design using a rectangular windows results in lower order filters. The same considerations apply to the reconstruction filter. In this case, the stop-band attenuation must be high since the added noise will be spread over a large bandwidth, particularly to the high frequencies for which our ears are most sensitive. The delay due to the anti-aliasing and reconstruction filters will usually be lower than the delay of the bandlimiting filter, because of the large transition band, even when the sampling frequency is not very high.

8. Simulation results

The algorithm proposed in Figs. 3–5 were all validated through numerical simulations. They all proved to work as expected even in the presence of secondary path modelling errors. The secondary paths were first modelled off-line in a classic system identification scenario [1]. The simulation results are presented in Figs. 6–8. The DSP sampling frequency was 4 kHz for Configurations I and II, and of 1 kHz for Configuration III. The reference signal was modelled as white noise. The secondary path and primary path were modelled as pure delays corresponding to fractional values of the DSP sampling time. In order to simulate these delays the analogue section was simulated in discrete time, but at a higher sampling rate. The anti-aliasing, reconstruction and bandlimiting filters were FIR filters designed using a Hamming window. This is not the best choice but is enough to demonstrate the effectiveness of the algorithms. The bandlimiting filter had a transition band from 400 to 500 Hz. The anti-aliasing and reconstruction filters had a larger transition band from 400 Hz to \( F_s/2 \). The step of the LMS algorithm was normalized [1] and \( \alpha \) was set to 0.3. Figs. 6–8 show the convergence curve of the

![Fig. 6. Convergence curve and noise level with and without ANC for Configuration I.](image-url)
algorithms in Configurations I–III. It can be seen that all the algorithms converge as expected. The figures also show the noise levels with and without ANC. It can be seen that the noise level is only reduced in the 0–400 Hz band, as desired, which implies that the anti-noise was in fact limited to 400 Hz.

The results for the sensitivity to secondary path modelling errors were tested in Configuration I, with $\alpha = 0.03$ and $F_S = 4$ kHz sampling frequency. For the sinusoidal reference the anti-noise filter cut frequency was set to 500 Hz. The maximum error for the delay of the cancellation path estimate, $d_M$, was measured for different anti-noise bandlimiting frequencies, $F_c$, and reference frequencies, $f$, for the case of sinusoidal input and compared to the theoretical predicted value, $d_M(T)$, derived from (44) and (46). Table 1 shows the results for a $1 \times 1$ system with a sinusoidal reference, and for a $2 \times 3 \times 2$ system with a white noise reference. The results are in close agreement with the theory.

9. Conclusion

In broadband ANC systems, limiting the anti-noise frequency results in large quiet zones, without the use of several error signals per anti-noise source. In this paper, techniques to do this for multichannel systems are presented. They all work as expected. This techniques increase the stability to secondary path modelling errors, allow the use of low-order and delay anti-aliasing and reconstruction filters, and finally, allow systems with more anti-noise sources than error sensors for better cancellation.
path inversion while still achieving large quiet zones. Maximum delay errors in the cancellation path estimate increase as the anti-noise bandlimiting frequency decreases.

References


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<th>$d_M$ (theory)</th>
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