ON THE $H_2$ OPTIMALITY OF SOME COMMON PI TUNING RULES

Shirley, P.O.*†, 1 Lemos, J.M.**†

* Universidade Aberta
R. Escola Politécnica 147, 1269-001 Lisboa, Portugal
** Instituto Superior Técnico, Portugal
† INESC-ID, R. Alves Redol 9, 1000-029 LISBOA, Portugal

Abstract: Common PI tuning rules such as the Ziegler-Nichols, Cohen-Coon and integral error like, are compared with the optimal PI obtained minimizing the $H_2$ norm of the closed loop system for a step disturbance at the system input. The optimal PI is computed through numerical optimization using a static output feedback framework. Results are compared on the basis of the error cost $J_e$ and the actuator cost $J_u$.

Keywords: Optimal control; $H_2$; Numerical optimization; PI tuning; Static Output Feedback .

1. INTRODUCTION

PID and PI are the most used controllers in the industry. With an intuitive working concept and a high performance to complexity structure ratio, they are hard to beat. Over the times, there have been proposed many simple tuning rules, usually assuming also a simple model for the process to be controlled, namely, the first order plus dead time model (FOPDT),

$$G_p(s) = \frac{K_p}{1 + T_a s} e^{-T_d s}$$  (1)

For this model, Table 1 shows six common PI tuning rules (Ho et al., 1995): Ziegler-Nichols (ZN), Cohen-Coon (CC), Integral Squared Error (ISE), Integral Absolute Error (IAE), Integral Time Squared Error (ITSE) and Integral Time Absolute Error (ITAE), where the later four are specially suited for disturbance rejection. In this paper, the PI controllers generated with these tuning rules are compared with the one obtained with the $H_2$ criterium of optimality for a step load disturbance in the input.

<table>
<thead>
<tr>
<th>Formulas</th>
<th>$K_c$</th>
<th>$T_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZN</td>
<td>$\frac{0.6 \times T_d}{K_p}$</td>
<td>$3T_d$</td>
</tr>
<tr>
<td>CC</td>
<td>$\frac{1}{K_p} \left( \frac{0.5 \times T_d}{T_p} + 0.082 \right)$</td>
<td>$\frac{3.33T_d + 0.33(T_d^2)}{1.72T_d^2}$</td>
</tr>
<tr>
<td>ISE</td>
<td>$\frac{2.86}{K_p} \left( \frac{T_p}{T_d} \right)^{-0.966}$</td>
<td>$\frac{T_d}{0.097} \left( \frac{T_d}{T_p} \right)$</td>
</tr>
<tr>
<td>IAE</td>
<td>$\frac{0.098}{K_p} \left( \frac{T_d}{T_p} \right)^{-0.966}$</td>
<td>$\frac{T_d}{0.097} \left( \frac{T_d}{T_p} \right)$</td>
</tr>
<tr>
<td>ITSE</td>
<td>$\frac{1.73}{K_p} \left( \frac{T_p}{T_d} \right)^{-0.945}$</td>
<td>$\frac{T_d}{0.538} \left( \frac{T_d}{T_p} \right)^{0.586}$</td>
</tr>
<tr>
<td>ITAE</td>
<td>$\frac{0.833}{K_p} \left( \frac{T_d}{T_p} \right)^{-0.977}$</td>
<td>$\frac{T_d}{0.668} \left( \frac{T_d}{T_p} \right)^{0.08}$</td>
</tr>
</tbody>
</table>

Table 1. Common PI tuning rules.

The structure of this paper is as follows. Section II describes the $H_2$ optimal methodology. Section III presents the comparisons for two example systems and the conclusions are given in section IV.
2. THE CONTROL SETUP

Consider the control setup shown in Figure 1, where \( w \in \mathbb{R}^{nw} \) is a disturbance input (such as tracking signals, measurable and non measurable disturbances), \( u \in \mathbb{R}^{nu} \) is the control input, \( z \in \mathbb{R}^{nz} \) is the controlled output (such as error and control signals) and \( y \in \mathbb{R}^{ny} \) is the measured output. The generalized plant \( G_0 \) includes the system to be controlled and the additional elements to generate the signals involved. It is assumed that \( G_0 \) does not have direct terms from \( w \) to \( z \) and \( y \). The \( F \) block is a matrix gain and implements the Static Output Feedback (SOF) (Levine and Athans, 1970) control law \( u = F y \). This law is not a limitation as a fixed order control problem can be recasted as a SOF control problem (Keel and Bhattacharyya, 1990).

The generalized system \( G_0 \) admits the following state space representation

\[
G_0 \leftrightarrow \begin{cases} \dot{z} = A x + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x + D_{22} u \end{cases}
\]

The closed loop system when \( u = F y \) is given by

\[
G_0(F) \leftrightarrow \begin{cases} \dot{z} = A_F x + B_1 w \\ z = C_1 F x \end{cases}
\]

\( P_F = (I - F D_{22})^{-1} \), \( P_{DF} = (I - D_{22} F)^{-1} \)

\( A_F = A + B_2 P_F F C_2, \quad C_{1F} = C_1 + D_{12} P_F F C_2 \)

Define the matrices \( W = \sum_{i=1}^{nw} w_i w_i' \geq 0 \), \( Q = Q' \geq 0 \), \( \delta w_i \) as \( w_i \delta(t) \) and the vector norm

\[
\|v(t)\|_{Q_2}^2 = \int_0^{+\infty} e^{-2\sigma r} v'(r) Q v(r) dr.
\]

The cost functional is the \( H_2 \) type norm

\[
J(F, \alpha) = \sum_{i=1}^{nw} \|G_0(F_i) \delta w_i\|_{Q_2}^2 = \text{tr}(B_1 X B_1^T) + \text{tr}(C_{1F} Y C_{1F}' Q)
\]

\( (A_F - \alpha I)Y + Y (A_F - \alpha I)' + B_1 W B_1' = 0 \)

For solving optimization problems with \( J(F, \alpha) \) with numerical techniques, the first order derivatives with respect to \( F \) are given by (Shirley and Lemos, 2001)

\[
\frac{\partial J(F, \alpha)}{\partial F} = 2(D_{12} Q C_{1F} + B_2 X Y C_{1F}' (7)
\]

The \( \alpha \) parameter provides a tool to deal with unstable systems. If an initial controller \( F_0 \) is not stabilizing, the idea is to start with a stabilizing \( \alpha \) and progressively decrease it as the numerical optimization proceeds and reach \( \alpha = 0 \).

3. RESULTS

Results are presented for two extreme cases of system 1, parameterized with \( K_P = 1 \), \( \tau = 1 \) and delay \( T \in \{0.1, 1\} \), or from another perspective, with ratios delay to time constant of \( T/\tau \in \{0.1, 1\} \). The generalized system in Figure 1 is built for a regulator problem with a step disturbance \( d \) applied at the system input, modelled as the impulse response of a perturbed integrator \( d = H d \delta d \),

\[
H_d(s) = \frac{1}{s + 10^{-3}} \simeq \frac{1}{s} \quad (8)
\]

since this pole is not controllable. For the generalized system, the perturbation is \( w = \delta d \) and the regulated variable \( z = [y_p, u_p]' \), where \( y_p \) and \( u_p \) are respectively the output and input of system 1. The time delay \( T \) is modelled with a fourth order Padé approximation. Figure 2 shows the initial step responses of system 1 for \( T = 0.1 \) and \( T = 1 \). Oscillations due to the time delay approximation are kept under 3% of the final value.

The cost functional 4 is parameterized with \( W = 1 \) and \( Q = Q_\epsilon + \rho Q_u \), where \( Q_\epsilon = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] \) and \( Q_u = \left[ \begin{array}{cc} 9 & 0 \\ 0 & 1 \end{array} \right] \). The error cost is given by \( J_e = J(Q_\epsilon) \) and the actuator cost by \( J_u = J(Q_u) \). The optimal \( H_2 \)-PI controller is computed for several values of \( \rho \), giving the Pareto optimal curve for the error and actuator cost. For comparison, the values of \( J_e \) and \( J_u \) are also computed for each controller obtained with the tuning rules of table 1.

Figure 3 shows the results for \( T/\tau = 0.1 \) and \( \rho \in [10^{-6}, 10^6] \). Values outside this interval do not produce larger values of \( J_e \) and \( J_u \). Values inside the interval are selected in order to obtain a smooth curve for graphical representation. As can be seen in the graph, all tuning rules produce near optimal values of the costs with the exception of ISE, which gives a too high actuator cost (but
good performance). The arrows show some values of $\rho$ along the curve.

Figure 3 shows the results for $T/\tau = 1$ and $\rho \in [10^{-6}, 10^4]$. Again, values of $\rho$ outside this interval do not extend the curve. The ZN controller yields the worst case with $(J_c, J_u) = (1.13, 1.68)$ and is not shown in the graph. The CC, IAE and ITAE controllers are also relatively far from the optimal curve. The only ones that remain close to the optimal curve are the ISE and ITSE controllers.

Looking for the results in Figures 3 and 4, ITSE comes out as the best PI controller from the $H_2$ point of view, as it remains on the optimal curve for the two extremal values of the $T/\tau$ ratio.

This paper analyzes the $H_2$ optimality of some common PI tuning rules for simple FOPTD models with delay to time constant ratio of 0.1 and 1. For a small ratio of 0.1 all tuning rules gave near optimal results except the ISE rule. For a larger ratio of 1, only the ISE and ITSE succeeded. As the only one to give optimal results for both ratios, the ITSE is the clear “winner” from the analyzed set.

4. CONCLUSIONS

5. REFERENCES


