Modeling for Control Design
of a Multi-Reach Water Canal

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Contents

Abstract .......................................................................................................................... 1

1 INTRODUCTION ........................................................................................................ 1
   1.1 Motivation ............................................................................................................. 1
   1.2 Literature review .................................................................................................. 2
   1.3 Main contributions ............................................................................................... 2

2 Canal description .......................................................................................................... 2

3 Linearization of the Saint-Venant equations ............................................................. 3
   3.1 Saint-Venant equations ....................................................................................... 3
   3.2 Linearization ....................................................................................................... 4
   3.3 Exact transfer function ....................................................................................... 8

4 Finite dimension models ............................................................................................ 10
   4.1 Poles computation ............................................................................................... 10
   4.2 Residues computation ....................................................................................... 11
   4.3 Rational transfer function .................................................................................. 11

5 Overall model ............................................................................................................... 15
   5.1 Gate model .......................................................................................................... 15
   5.2 Overall model ...................................................................................................... 15

6 Multi-reach canal ......................................................................................................... 18

7 CONCLUSION .............................................................................................................. 22
Abstract

This report addresses the problem of obtaining reduced complexity models of multi-reach water delivery canals that are suitable for robust and linear parameter varying (LPV) control design. In the first stage, by applying a method known from the literature, a finite dimensional rational transfer function of \textit{a priori} defined order is obtained for each canal reach by linearizing the Saint-Venant equations. Then, by using block diagrams algebra, these different models are combined with linearized gate models in order to obtain the overall canal model. In what concerns the control design objectives, this approach has the advantages of providing a model with prescribed order and to quantify the high frequency uncertainty due to model approximation.

1 INTRODUCTION

1.1 Motivation

Water delivery canals are large-scale systems made of reaches that are interconnected by gates that control the water flow on them [1]. The fact that their dynamics depends on space as well as on time leads to the use of models that rely on partial differential equations (PDE). In the case of a single canal reach, the water flow, under shallow conditions, is modeled by the Saint-Venant equations [2], a pair of nonlinear PDE that embed mass and momentum conservation.

Given an operating point defined by an average water flow and level, the Saint-Venant equations can be linearized to yield a model that is suitable for controller design. However, the following issues have to be considered:

- The linearization \textit{per se} yields an infinite dimensional model that for practical purposes has to be approximated by a rational transfer function with a prescribed order;
- The transfer functions that relate the inlet and outlet flows with the upstream and downstream levels in each canal reach must be combined with the linearized gate models to yield an overall linear canal model that relates the gate positions (manipulated inputs) and the side-take water extractions (disturbance inputs) with the downstream level of each canal reach ("process" outputs);
- The models depend on the operating point; the operation around different flow and level equilibrium values leads to different linearized models.

In this article a modeling procedure that addresses these three issues is illustrated in a case study.

Considering that the models to obtain are to be used for controller design, following an approach based on the physical modeling has the advantage of providing a model with \textit{a priori} prescribed order and to quantify the high frequency uncertainty due to
model approximation, thereby providing a natural framework for robust controller design. Furthermore, the method provides a pencil on models that correspond to different flow and level equilibria and that is suitable for use with approaches based on linear parameter value (LPV) design [3, 4].

1.2 Literature review

Modeling of water delivery canals can be done by either data driven or physical model driven approaches [5]. Data driven methods consist of system identification methods that are used to fit a linear model with an \textit{a priori} selected structure, such as an ARX model, to canal data [6, 7]. Data driven methods have the advantage of allowing to impose \textit{a priori} the model structure. Their main drawbacks consist of requiring costly experiments that must be performed in the canal (implying that the canal has been already built and is available for experimentation), and on the fact that a linearized model for each operating condition requires different experimental data.

The alternative is to use an approach that relies on basic physical conservation laws, and leads to the Saint-Venant equations for canal reaches and the Bernoulli / energy conservation law to relate the gate position with the flow across each gate, and then to linearize and approximate the infinite dimensional models by finite dimensional ones using residues computation in the Laplace-variable domain [2, 8]. In [9], a case study using this methodology and performed in the same canal considered in this paper is presented. However, in [9] the canal has been configured with all its gates fully opened, except the last one, thereby considering only one reach, while in the present paper the multi-reach (3 reaches) situation is considered.

1.3 Main contributions

The main contribution of this study consists of a case study of physically based canal modeling. The fact that a multi-reach canal configuration is considered implies that the study also addresses the manipulation of the basic transfer functions (canal reaches, gates) to obtain an overall canal system model.

2 Canal description

The canal considered in this study is the experimental facility of Núcleo de Hidráulica e Controlo de Canais (NuHCC) of the University of Évora, in Portugal. This multi-reach canal is composed of three reaches with 35 m long and a fourth reach with 36 m long, with a total length of 141 m. The automatic canal has a trapezoidal cross section with bottom width 0.15 m, sides slope 1:0.15 (V:H) and depth 0.9 m. The canal was designed for a maximum flow of 0.09 m$^3$/s and the longitudinal bed slope is $1.5 \times 10^{-3}$. The reaches are interconnected with three automated undershot gates and at the end of the last reach an overshot gate is placed. The water level is measured at the downstream of
each reach by a floating sensor. The water flows into the canal by gravity and the input
flow is imposed by a valve driven by a flow controller.

3 Linearization of the Saint-Venant equations

The computation of the transfer function for the incremental response of the linearized
Saint-Venant equations around an equilibrium defined by constant values of the flow
and the level have been considered for several authors [2,8,10]. We explain the main
steps hereafter for the sake of clarity. First, the Saint-Venant equations are linearized,
using jacobian linearization, around an equilibrium point. The Laplace transform with
respect to time is applied to the linear PDE satisfied by the increments, resulting in a
linear ordinary differential equation (ODE) in the space variable ($x$). The solution of
this ODE using the formula of variation of constants [11] results in a transfer function
that relates the input and output reach flows (for $x = 0$ and $x = L$, the length of the
canal reach) with the upstream ($x = 0$) and the downstream ($x = L$) water levels on
the reach. In order to obtain a rational transfer function with a finite number of poles,
the dominant poles (that correspond to the poles at the lowest frequency range) are
computed. The computation of the associated residues, that can be accomplished with
l’Hôpital formula, finally allows to obtain a finite dimension rational transfer function
obtained as a sum of first order transfer functions, each associated with one pole. The
details of this procedure are described hereafter.

3.1 Saint-Venant equations

The water flow under shallow water conditions is modeled by the nonlinear Saint-Venant
equations [2]. These nonlinear partial differential equations are obtained from the phys-
ical principles of mass conservation and momentum conservation and are defined by

$$\frac{\partial A(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = 0,$$

(1)

$$\frac{\partial Q(x,t)}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2(x,t)}{A(x,t)} \right) + g A(x,t) \left( \frac{\partial Y(x,t)}{\partial x} + S_f(x,t) - S_b(x) \right) = 0,$$

(2)

where $Q(x,t)$ (m$^3$/s) is the discharge across section $A(x,t)$ (m$^2$), which is the wetted
area, $g$ (m/s$^2$) is the gravitational acceleration, $Y(x,t)$ (m) is the water depth, $S_f(x,t)$ is
the friction slope, $S_b(x)$ (m/m) is the longitudinal bed slope, $t$ is continuous time and $x$
is the abscissa along the reach. The expression (1) is referred as the continuity equation
and (2) is referred as the momentum equation. The cross section is described by

$$A(x,t) = T(x,t) Y(x,t),$$

(3)

where $T(x,t)$ (m) is the top width of the wetted cross section. The discharge is described
by

$$Q(x,t) = A(x,t) V(x,t),$$

(4)
where \( V(x,t) \) (m/s) is the water average velocity on section \( A(x,t) \). The friction slope is modeled by the Manning-Strickler formula

\[
S_f(x,t) = \frac{Q(x,t)^2 n^2}{A(x,t)^2 R(x,t)^{4/3}},
\]

where \( n \) (m\(^{-1/3}\)s) is the Manning coefficient and \( R(x,t) \) (m) is the hydraulic radius, described by \( R(x,t) = A(x,t)/P(x,t) \), with \( P(x,t) \) being the perimeter of the wetted area. The initial conditions of the PDEs are \( Q(x,0) \) and \( Y(x,0) \) and the boundary conditions considered are \( Q(0,t) \) and \( Q(L,t) \), where \( L \) (m) is the length of the reach.

### 3.2 Linearization

The nonlinear Saint-Venant equations are linearized by jacobian linearization, which for a function \( f(Q,Y) \) is the first order terms of its Taylor expansion around the equilibrium point \((Q_0,Y_0)\), such as

\[
f(Q,Y) - f(Q_0,Y_0) \approx \frac{\partial f}{\partial Q}_{(Q_0,Y_0)} (Q - Q_0) + \frac{\partial f}{\partial Y}_{(Q_0,Y_0)} (Y - Y_0),
\]

with equilibrium values of flow \( Q_0 \) and level \( Y_0 \). Considering \( q(x,t) \) and \( y(x,t) \) to be the deviations from the equilibrium values, such that \( Q(x,t) = Q_0(x) + q(x,t) \) and \( Y(x,t) = Y_0(x) + y(x,t) \), the linearization is applied to each term of the Saint-Venant equations. In equilibrium the top width of the wetted section is \( T(x,t) = T_0(x) \). From (3), the first term of the continuity equation (1) is linearized as follows

\[
\frac{\partial A}{\partial t} \approx \frac{\partial}{\partial t}(TY) = \frac{\partial}{\partial t}(T_0 Y - T_0 Y_0) = \frac{\partial}{\partial t}\left(\frac{d(T_0 Y)}{dY}\right)_{(Q_0,Y_0)} y = T_0 \frac{\partial y}{\partial t},
\]

and the second term is

\[
\frac{\partial Q}{\partial x} \approx \frac{\partial}{\partial x}\left(\frac{dQ}{dQ}\right)_{(Q_0,Y_0)} q = \frac{\partial q}{\partial x},
\]

as well as the first term of the momentum equation (2),

\[
\frac{\partial Q}{\partial t} \approx \frac{\partial q}{\partial t}.
\]

In the same way, the second term of (2) is linearized as such

\[
\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) \approx \frac{\partial}{\partial x}\left(\frac{Q^2}{A} - \frac{Q_0^2}{A_0}\right) = \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)_{(Q_0,Y_0)} q + \frac{\partial}{\partial Y}\left(\frac{Q^2}{A}\right)_{(Q_0,Y_0)} y
\]

\[
= \frac{\partial}{\partial x}\left(2Q_0 q - \frac{Q_0^2}{A_0} \frac{\partial A_0}{\partial Y} y\right)
\]

\[
= \frac{\partial}{\partial x}\left(2V_0 q - V_0^2 T_0 y\right),
\]

\[
(10)
\]
with \( V_0 \) the water velocity in equilibrium. Applying the derivative with respect to \( x \) to the argument, the last equality of (10) becomes

\[
\frac{\partial}{\partial x} \left( 2V_0 q - V_0^2 T_0 y \right) = 2 \frac{dV_0}{dx} q + 2V_0 \frac{\partial q}{\partial x} - \left( 2V_0 T_0 \frac{dV_0}{dx} + V_0^2 \frac{dT_0}{dx} \right) y - V_0^2 T_0 \frac{\partial y}{\partial x}.
\]

(11)

Since the linearization is performed for an equilibrium value of \( Q_0 \) and that in steady state flow the continuity equation is reduced to \( dQ_0/dx = 0 \), the last equality of (11) results from the following derivative

\[
\frac{dV_0}{dx} = 1 + \frac{dQ_0}{dx} - Q_0 \frac{dA_0}{dx} = - \frac{Q_0 T_0}{A_0} \frac{dY_0}{dx} = - \frac{V_0 T_0}{A_0} \frac{dY_0}{dx}.
\]

(12)

The third term of (2) is linearized as follows

\[
g A \frac{\partial Y}{\partial x} = g \left( T_0 Y \frac{\partial Y}{\partial x} + T_0 Y_0 \frac{dY_0}{dx} \right) = g \frac{d}{dY} \left( T_0 Y \frac{\partial Y}{\partial x} \right) \bigg|_{Y_0} y
\]

\[
= g T_0 \frac{dY_0}{dx} y + g T_0 Y_0 \frac{\partial y}{\partial x}.
\]

(13)

The fourth term (2) results in the following expression

\[
g A (S_f - S_b) = g A (S_f - S_b) - g A_0 (S_{f0} - S_b)
\]

\[
= g \frac{\partial}{\partial Q} \left( A (S_f - S_b) \right) \bigg|_{(Q_0,Y_0)} q + g \frac{\partial}{\partial Y} \left( A (S_f - S_b) \right) \bigg|_{(Q_0,Y_0)} y
\]

\[
= g A_0 \frac{\partial f}{\partial Q} \bigg|_{(Q_0,Y_0)} q + g \left( A_0 \frac{\partial f}{\partial Y} \bigg|_{(Q_0,Y_0)} + \frac{dA}{dY} \bigg|_{(Q_0,Y_0)} (S_{f0} - S_b) \right) y.
\]

(14)

The last term of (14) is directly obtained as \( g T_0 (S_{f0} - S_b) y \), with \( S_{f0}(x) \) the friction slope in equilibrium, and recalling (5) the expression of the derivatives of (14) are

\[
\frac{\partial S_f}{\partial Q} \bigg|_{(Q_0,Y_0)} = \frac{\partial}{\partial Q} \left( \frac{Q^2 n^2}{A^2 R^{4/3}} \right) \bigg|_{(Q_0,Y_0)} = 2 \frac{Q_0 n^2}{A_0^2 R_0^{4/3}} = 2 \frac{S_{f0}}{Q_0},
\]

(15)

\[
\frac{\partial S_f}{\partial Y} \bigg|_{(Q_0,Y_0)} = \frac{\partial}{\partial Y} \left( \frac{Q^2 n^2}{A^2 R^{4/3}} \right) \bigg|_{(Q_0,Y_0)} = -2 \frac{Q_0^2 n^2}{A_0^2 R_0^{4/3}} \frac{\partial A_0}{\partial Y} - \frac{4}{3} \frac{Q_0^2 n^2}{A_0^2 R_0^{7/3}} \frac{\partial R_0}{\partial Y}
\]

\[
= -2 \frac{S_{f0} T_0}{A_0} - \frac{4}{3} \frac{S_{f0}}{A_0} \left( \frac{T_0}{A_0} - \frac{1}{P_0} \frac{\partial P_0}{\partial Y} \right).
\]

(16)

The last equality of (16) results from the fact that the derivative of \( R_0 \) with respect to \( Y \) is as follows

\[
\frac{\partial R_0}{\partial Y} = \frac{T_0}{P_0} - \frac{A_0}{P_0^2} \frac{\partial P_0}{\partial Y}.
\]

(17)
From (14)-(16) the fourth term of (2) is

$$g A(S_f - S_b) = 2g \frac{A_0 S_{f_0}}{Q_0} q - g A_0 S_{f_0} \left[ \frac{T_0}{A_0} \left( \frac{7}{3} + S_b \right) - \frac{4}{3P_0 \partial Y} \right] y.$$

By gathering the linearized terms (7) and (8) of the Saint-Venant equations, the linearized continuity equation is

$$T_0 \frac{\partial y}{\partial t} + \frac{\partial q}{\partial x} = 0. \quad (19)$$

The momentum equation results from grouping (9), (11), (13) and (18), such as

$$\frac{\partial q}{\partial t} + 2 \left( g \frac{A_0 S_{f_0}}{Q_0} - \frac{V_0 T_0 dY_0}{A_0} \right) q + 2V_0 \frac{\partial q}{\partial x}$$

$$+ \left( 2V_0^2 T_0 \frac{dY_0}{dx} - V_0^2 \frac{dT_0}{dx} + gT_0 \frac{dY_0}{dx} - g A_0 S_{f_0} \left[ \frac{T_0}{A_0} \left( \frac{7}{3} + S_b \right) - \frac{4}{3P_0 \partial Y} \right] \right) y$$

$$+ (g A_0 - V_0^2 T_0) \frac{\partial y}{\partial x} = 0. \quad (20)$$

Introducing the Froude number as

$$F_0^2 = \frac{V_0^2 T_0}{gA_0}, \quad (21)$$

the factor of $q$ in (20) becomes

$$2g \left( \frac{S_{f_0}}{V_0} - \frac{V_0 T_0}{gA_0} \right) = 2g \left( S_{f_0} - F_0^2 \frac{dY_0}{dx} \right), \quad (22)$$

and the factor of $y$ becomes

$$gT_0 \left( 2F_0^2 \frac{dY_0}{dx} + \frac{dY_0}{dx} - \frac{7}{3} S_{f_0} - \frac{4}{3} A_0 S_{f_0} \frac{\partial P_0}{\partial Y} \right) - V_0^2 \frac{dT_0}{dx}$$

$$= -V_0^2 \frac{dT_0}{dx} - gT_0 \left[ \frac{7}{3} - \frac{4}{3} A_0 \frac{\partial P_0}{\partial Y} \right] S_{f_0} + s_b - (1 + 2F_0^2) \frac{dY_0}{dx}. \quad (23)$$

Considering the celerity $C(x,t)$ defined as

$$C(x,t) = \sqrt{gY(x,t)} \quad (24)$$

and $C_0(x)$ its equilibrium value, the factor of $\partial y$/\partial x i (20) is $T_0 (C_0^2 - V_0^2)$ and, from (22) and (23), the linearized momentum equation is described by

$$\frac{\partial q}{\partial t} + 2V_0 \frac{\partial q}{\partial x} + \epsilon q + (C_0^2 - V_0^2) T_0 \frac{\partial y}{\partial x} - \gamma y = 0, \quad (25)$$
where the parameters $\epsilon$ and $\gamma$ are given by

$$
\epsilon = \frac{2g}{V_0} \left( S_{f_0} - F_0^2 \frac{dY_0}{dx} \right),
$$

$$
\gamma = V_0^2 \frac{dT_0}{dx} + gT_0 \left[ \kappa S_{f_0} + S_b - (1 + 2F_0^2) \frac{dY_0}{dx} \right],
$$

with

$$
\kappa = \frac{7}{3} - \frac{4}{3} A_0 \frac{\partial P_0}{\partial Y}.
$$

The expression of $S_{f_0}$ is deduced assuming the Saint-Venant equations (1,2) in steady state flow, which means that $\partial A/\partial t = 0$ and $\partial Q/\partial t = 0$. The continuity equation (1) in equilibrium becomes

$$
\frac{dQ_0}{dx} = 0,
$$

and the momentum equation (2) becomes

$$
\frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + g A \left( \frac{\partial Y}{\partial x} + S_f - S_b \right) = 0,
$$

$$
\frac{2}{A} \frac{\partial Q}{\partial x} - \frac{Q^2}{A^2} \frac{\partial A}{\partial x} + g A \left( \frac{\partial Y}{\partial x} + S_f - S_b \right) = 0.
$$

In equilibrium (31) is

$$
2 \frac{\partial Q_0}{A} \frac{dY_0}{dx} - \frac{Q_0^2 T_0}{A_0^2} \frac{dY_0}{dx} + g A_0 \left( \frac{dY_0}{dx} + S_{f_0} - S_b \right) = 0,
$$

$$
-V_0^2 T_0 \frac{dY_0}{dx} + C_0^2 T_0 \left( \frac{dY_0}{dx} + S_{f_0} - S_b \right) = 0,
$$

$$
\frac{dY_0}{dx} = \frac{S_b - S_{f_0}}{1 - F_0^2}.
$$

From (34) the expression of $S_{f_0}$ is

$$
S_{f_0} = S_b (1 - F_0^2) \frac{dY_0}{dx}.
$$

Therefore, with (35), the expressions for $\epsilon$ and $\gamma$ become

$$
\epsilon = \frac{2g}{V_0} (S_b - \frac{dY_0}{dx}),
$$

$$
\gamma = V_0^2 \frac{dT_0}{dx} + gT_0 \left[ (1 + \kappa) S_b - (\kappa + 1 - (\kappa - 2) F_0^2) \frac{dY_0}{dx} \right].
$$

Let us assume a constant cross section along the canal implying that $Y_0$ and $T_0$ are constant with respect to $x$ and, therefore, the flow is considered uniform, then the expressions for $\epsilon$ and $\gamma$ are

$$
\epsilon = \frac{2gS_b}{V_0},
$$

$$
\gamma = g(1 + \kappa)S_b.
$$
3.3 Exact transfer function

Defining the state vector \( \xi(x,t) = [T_0 y \; q]^T \), and \( \alpha(x) \) and \( \beta(x) \) such that

\[
\begin{align*}
\alpha(x) &= C_0(x) + V_0(x), \\
\beta(x) &= C_0(x) - V_0(x),
\end{align*}
\]

(40, 41)

the linearized Saint-Venant equations (19, 25) are written in the form

\[
\frac{\partial \xi}{\partial t} + \mathbf{A} \frac{\partial \xi}{\partial x} + \mathbf{B} \xi = 0,
\]

(42)

where \( \mathbf{A}(x) \) and \( \mathbf{B}(x) \) are matrices defined by

\[
\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \alpha \beta & -\beta \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ -\gamma & \epsilon \end{bmatrix}.
\]

(43)

Let one consider the Laplace transform of a function \( f \), \( \mathcal{L}\{ f(t) \} = \hat{f}(s) \), with \( s \) the Laplace variable, and the following property

\[
\mathcal{L}\left\{ \frac{\partial f(t)}{\partial t} \right\} = s \hat{f}(s) - f(0).
\]

(44)

Applying (44) to the time derivative of the linear partial differential equation (42), the Saint-Venant equations become

\[
\frac{\partial}{\partial x} \hat{\xi}(x,s) = \tilde{\mathbf{A}}(s) \hat{\xi}(x,s) + \tilde{\mathbf{B}}(s) \xi(x,0),
\]

(45)

where \( \tilde{\mathbf{A}}(s) = -\mathbf{A}^{-1}(s \mathbf{I} + \mathbf{B}) \), with \( \mathbf{I} \) denoting the identity matrix, \( \tilde{\mathbf{B}}(s) = \mathbf{A}^{-1} \), and \( \hat{\xi} \) referring to the state vector in the Laplace domain. Matrix \( \tilde{\mathbf{A}}(s) \) and \( \tilde{\mathbf{B}}(s) \) are given by

\[
\tilde{\mathbf{A}} = \begin{bmatrix} \frac{(\alpha-\beta)s+\gamma}{\alpha^2} & -s & 1 \\ \frac{s-\gamma}{\alpha^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} \frac{(\alpha-\beta)s+\gamma}{\alpha^2} & 1 \\ 0 & -s & 0 \end{bmatrix},
\]

(46)

and the eigenvalues \( \lambda_1(s) \) and \( \lambda_2(s) \) of \( \tilde{\mathbf{A}}(s) \), which are the solution of \( \det(\lambda \mathbf{I} - \tilde{\mathbf{A}}(s) = 0) \), are given by

\[
\lambda_1(s) = \frac{(\alpha-\beta)s+\gamma-\sqrt{\sigma(s)}}{2\alpha\beta},
\]

(47)

\[
\lambda_2(s) = \frac{(\alpha-\beta)s+\gamma+\sqrt{\sigma(s)}}{2\alpha\beta},
\]

(48)

with

\[
\sigma(s) = (\alpha + \beta)^2 s^2 + 2[(\alpha-\beta)\gamma + 2\alpha\beta \epsilon] s + \gamma^2.
\]

(49)
After diagonalizing the matrix $\tilde{A}(s)$, the solution of the ODE (45) is
\[
\dot{\xi}(x, s) = \Phi(x, s)\dot{\xi}(0, s) + \xi_0(x, s),
\] (50)
where
\[
\xi_0(x, s) = \Phi(x, s) \int_0^x \Phi(\nu, s)^{-1} \tilde{B}(s) \xi(\nu, 0) d\nu,
\] (51)
as the initial conditions and where $\Phi$ is the transition matrix defined by
\[
\Phi(x, s) = X(s) - 1 e^{D(s)x} X(s),
\] (52)
where $D$ is a diagonal matrix whose elements are the eigenvalues $\lambda_1$ and $\lambda_2$ and $X$ contains the eigenvectors of $\Phi(x, s)$ and is explicitly defined as
\[
X = \begin{bmatrix} \frac{s}{\lambda_2} & 1 \\ \frac{s}{\lambda_1} & 1 \end{bmatrix}.
\] (53)
Assuming zero initial conditions, $\xi_0 = 0$, the incremental system (50) is given by
\[
\begin{bmatrix} T_0 y(x, s) \\ q(x, s) \end{bmatrix} = \begin{bmatrix} \phi_{11}(x, s) & \phi_{12}(x, s) \\ \phi_{21}(x, s) & \phi_{22}(x, s) \end{bmatrix} \begin{bmatrix} T_0 y(0, s) \\ q(0, s) \end{bmatrix},
\] (54)
where $\phi_{ij}$, with $i, j = 1, 2$, are the elements of the state transition matrix $\Phi$, which are given by
\[
\begin{align*}
\phi_{11} &= \frac{\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}}{\lambda_1 - \lambda_2}, \\
\phi_{12} &= \frac{\lambda_1 \lambda_2 (e^{\lambda_1 x} - e^{\lambda_2 x})}{s(\lambda_1 - \lambda_2)}, \\
\phi_{21} &= \frac{s(e^{\lambda_2 x} - e^{\lambda_1 x})}{\lambda_1 - \lambda_2}, \\
\phi_{22} &= \frac{\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}}{\lambda_1 - \lambda_2}.
\end{align*}
\] (55) - (58)
Reordering the system (54), the input-output system that describes the water levels upstream and downstream of the reach from the upstream and downstream discharges is
\[
\begin{bmatrix} y(0, s) \\ y(x, s) \end{bmatrix} = P(s) \begin{bmatrix} q(0, s) \\ q(x, s) \end{bmatrix},
\] (59)
where $P(s)$ is the input-output transfer matrix and the transfer functions $p_{ij}(s)$, with
\(i, j = 1, 2\), which are the elements of \(P(s)\), are

\[
\begin{align*}
    p_{11}(s) &= \frac{\lambda_2(s)e^{\lambda_1(s)x} - \lambda_1(s)e^{\lambda_2(s)x}}{sT_0(e^{\lambda_2(s)x} - e^{\lambda_1(s)x})}, \\
p_{12}(s) &= \frac{\lambda_1(s) - \lambda_2(s)}{sT_0(e^{\lambda_2(s)x} - e^{\lambda_1(s)x})}, \\
p_{21}(s) &= \frac{(\lambda_2(s) - \lambda_1(s))e^{(\lambda_1(s) + \lambda_2(s))x}}{sT_0(e^{\lambda_2(s)x} - e^{\lambda_1(s)x})}, \\
p_{22}(s) &= \frac{\lambda_1(s)e^{\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}}{sT_0(e^{\lambda_2(s)x} - e^{\lambda_1(s)x})}.
\end{align*}
\]  

The input-output model (59-63) is an infinite dimensional model that has been described by several studies [2,8,10].

4 Finite dimension models

In order to achieve a model suitable for controller design, a finite dimension model is deduced, from the input-output model (59).

4.1 Poles computation

First, the explicit expression of the poles of the transition matrix \(\Phi\) is deduced. From (60-63), the characteristic equation of \(\Phi\) is seen to be

\[
s(e^{\lambda_2(s)L} - e^{\lambda_1(s)L}) = 0,
\]

which leads to a pole at the origin \((p_0 = 0)\), the other poles being obtained by

\[
\lambda_2 - \lambda_1 = \frac{2k\pi j}{L},
\]

with \(k \in \mathbb{N}^*\). From (47-49) and (65) the poles \(p_{\pm k}\) are given by the expression

\[
p_{\pm k} = -\frac{(\alpha - \beta)\gamma + 2\alpha\beta\epsilon}{(\alpha + \beta)^2} \pm \frac{2\alpha^2\beta^2}{(\alpha + \beta)^2} \sqrt{\Delta(k)},
\]

with

\[
\Delta(k) = \frac{(\alpha + \beta)^2}{\alpha^2\beta^2} \left[ \frac{(\alpha\epsilon - \gamma)(\beta\epsilon + \gamma)}{\alpha\beta(\alpha + \beta)^2} - \frac{k^2\pi^2}{L^2} \right].
\]

If the poles are complex conjugate ones, the subindex \(\pm k\) refers to the pole computed for \(+k\) and its conjugate for \(-k\). This index stands also for the real poles and, in this case, \(p_{+k}\) and \(p_{-k}\) are not complex conjugate and have different values.

In Fig. 1 is plotted the time constant of each pole, showing that it decreases as the number of poles increases.
4.2 Residues computation

For a generic meromorphic function $f(s)$, the residue $r_{\pm k}$, which is proportional to the contour integral of the function along a path enclosing a singularity $p_{\pm k}$, is computed by

$$r_{\pm k} = \lim_{s \to p_{\pm k}} (s - p_{\pm k}) f(s).$$

(68)

If $f$ is such that $f(s) = b(s)/(s a(s))$, and the derivative of $a(s)$ with respect to $s$, $a'(s)$, is nonzero, by the l’Hôpital’s rule (68) can be written as

$$r_{\pm k} = \lim_{s \to p_{\pm k}} \frac{b(s)}{a(s) + s a'(s)}.$$ 

(69)

The value of the residues of the model is computed by taking $f(s) = p_{ij}(s)$ for the different transfer functions (60-63), for each pole.

In Fig. 2 is plotted the residue of each pole, showing that it decreases as the number of conjugate poles increases.

4.3 Rational transfer function

With the explicit expression of the poles and of the respective residues, the model approximation is computed by

$$p_{ij}(s) = \sum_{k=0}^{n} \frac{r_{\pm k}}{s - p_{\pm k}},$$

(70)

where $n$ is the a priori selected order of the rational transfer function that approximates (59). For simplicity, we use the same symbol for $p_{ij}(s)$ and its finite dimension rational approximation given by (70).

Fig. 3 to Fig. 6 shows the frequency response of the exact infinite dimensional transfer function $p_{ij}(s)$, computed from (60-63), for $s = j\omega$, at frequency $\omega$, and four finite

Figure 1: Time constant related to each pole $p_k$. 

[Graph showing time constant related to each pole $p_k$.]
dimension rational transfer functions, computed from (70) for different values of $n$ (1, 3, 5 and 9). The finite dimensional approximate transfer functions show an adequate approximation of the model for the low-frequency region. For the model with only one pole at the origin, the model shows the integral behavior with a decay of 20 dB/decade. The higher order models show the approximation of the multiple resonance behavior up to a frequency that increases with the model order.

The pole-zero map of the transfer functions $p_{ij}$ of order 1, 3 and 5 are presented in Fig. 7 to Fig. 9.
Figure 4: Frequency response of the transfer functions $p_{12}$: finite dimension models of 1st, 3rd, 5th and 9th order, compared with the exact transfer function.

Figure 5: Frequency response of the transfer functions $p_{21}$: finite dimension models of 1st, 3rd, 5th and 9th order, compared with the exact transfer function.
Figure 6: Frequency response of the transfer functions $p_{22}$: finite dimension models of 1st, 3rd, 5th and 9th order, compared with the exact transfer function.

Figure 7: Poles ($p_k$) and zeros ($z_k$) of the transfer functions $p_{ij}$ of 1st order.

Figure 8: Poles ($p_k$) and zeros ($z_k$) of the transfer functions $p_{ij}$ of 3rd order.
5 Overall model

In this section an overall model for a multi-reach canal, that encompasses the model of the reaches and the model of the gates located at the downstream end of each reach is shown. This model is obtained by block diagram algebra from the blocks corresponding to the reach transfer functions and the linearized gate models, which interconnections are defined by the boundary conditions.

5.1 Gate model

The discharge between reaches are controlled by the gates located at the downstream end of the reach. The discharge through the gate is modeled by

\[ Q(s) = C_d A_u(s) \sqrt{2g(Y_l(s) - Y_r(s))}, \]  

(71)

where \( C_d \) is the coefficient discharge of the gates, \( A_u(s) = B_w U(s) \) is the area of gate cross section of bottom width \( B_w \) and gate position \( U \), and \( Y_l \) and \( Y_r \) are the gate upstream (left) and downstream (right) water levels, respectively. Performing the jacobian linearization of the gate model (71) at the equilibrium point \( \bar{Q} = Q(\bar{U}, \bar{Y}_l, \bar{Y}_r) \), with \( U = \bar{U} + u, \ Y_l = \bar{Y}_l + y_l \) and \( Y_r = \bar{Y}_r + y_r \), where \( u, y_l \) and \( y_r \) are incremental values of gate position and of the water levels, the incremental model of the gate becomes

\[ q = \varphi u + \delta (y_l - y_r), \]  

(72)

where \( \varphi \) and \( \delta \) are constant values that result from the linearization.

5.2 Overall model

Considering the input-output model (59) and the linearized gate model (72), the overall linear model of the reach is

\[ y_l = \frac{p_{21}}{1 - \delta p_{22}} q_0 + \frac{\varphi p_{22}}{1 - \delta p_{22}} u - \frac{\delta p_{22}}{1 - \delta p_{22}} y_r \]  

(73)
where $q_0$ is the deviation from the equilibrium from $Q_0$.

Fig. 10 depicts the block diagram representation of reach number $i$ with the notation of (72-73) changed to include the reach index. Here $y_i$ is replaced by $y_{id}$ and $y_r$ by $y_{iu}$. The whole canal model (shown in Appendix (Fig. 19)) is obtained by interconnecting block diagrams as in Fig. 10, one for each reach, and the input flow for $i = 1$ to be a given function that for the equilibrium flow is $q_0 = 0$.

Fig. 11 shows the frequency response of the transfer function of only one reach, from the gate position $u$ to the water level upstream to the gate, which is also the reach downstream. The model is a low-pass function with the low frequency phase at $180^\circ$, that is consistent with the fact that the water level decreases when the gate position increases. Fig. 11 also shows the finite dimensional models of four different orders, showing an adequate approximation to the exact transfer function in the low frequency range.

The time response of Fig. 12 shows the step response of the downstream water level of the reach for the four finite dimensional models.

![Figure 10: Block diagram of reach $i$ of the canal.](image-url)
Figure 11: Frequency response: finite dimension models of the water level as a function of the gate level of 1st, 3rd, 5th and 9th order, compared with the exact transfer function.

Figure 12: Time response to a negative step of amplitude $-0.1$ m applied to the gate at $t = 50$ s using finite dimension models of the water level as a function of the gate level of 1st, 3rd, 5th and 9th order.
6 Multi-reach canal

Considering a canal structure with three reaches as depicted in Fig. 13, the water level is modeled by the incremental relation

\[
\begin{bmatrix}
  y_{1d} \\
  y_{2d} \\
  y_{3d}
\end{bmatrix} = G(s) \begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix} + H(s) q_0, \quad (74)
\]

where \( G(s) \) and \( H(s) \), which are matrices composed by transfer functions \( g_{ij} \) and \( h_i \), with \( i, j = 1, 2, 3 \), are obtained by block diagram algebra, assuming a model such as shown in Fig. 19 (in Appendix or, equivalently, Fig. 20).

The multi-reach model relates the downstream water level increment of each reach with all the three gate positions increments and with the input discharge increment \( q_0 \).

The frequency response of the transfer functions \( g_{ij} \) of order 3 is plotted in Fig. 14 for a 3-reach canal with the following equilibrium values of the water level on each reach \( y_1 = 0.7 \) m, \( y_2 = 0.58 \) m, \( y_3 = 0.51 \) m, and \( y_4 = 0.47 \) m, with equilibrium values of the gate positions as \( u_1 = u_2 = 0.1 \) m and \( u_3 = 0.08 \) m, and for an equilibrium flow of 18 l/s.

The time responses to step increments on the gate levels are plotted in Fig. 15, with the selected order of 3.

The effect of increasing the equilibrium value of the flow is also shown in Fig. 15, which influences the dynamics of the system by increasing its speed of response.

The effect of the equilibrium values of the water levels on the time response of the system is plotted in Fig. 16, showing that the speed of response of the system increases with an increase of the water level drop.

Fig. 17 provides an overall view of the effect of the equilibrium values of the flow and of the water level on the dominant time constant \( \tau_D \), showing that it decreases when the flow increases and when the water level decreases.

The finite model of order three is used in Fig. 18 to make a simulation of the response to a square signal that is compared with experimental results for the first two canal
Figure 14: Frequency response of the transfer functions $g_{ij}$, from the input $u_i$ to the output $y_j$. Magnitude in dB and phase in degrees.

Figure 15: Time response: incremental values of the downstream water levels simulated in response to a step of $+0.1$ m on the 1st gate position (above) and on the 2nd gate position (below), for discharge equilibriums of 0.018 m/s and 0.04 m/s.
Figure 16: Time response: incremental values of the downstream water levels simulated in response to a step of +0.1 m on the 1st gate position (above) and on the 2nd gate position (bellow), for equilibrium values of the water level of 0.2 m apart.

Figure 17: Dominant time constant $\tau_D$ as a function of the flow and water level equilibrium values.
reaches. It is stressed that this comparison is made using only nominal values of the physical parameters of the canal, without using identification methods to find the best fitting.

Figure 18: Comparison of the model time response with the experimental data incremental values of the downstream water level $y_d$ simulated with a 3$^{rd}$ order model and with the gate position manipulations $u$; (a) and (b) correspond to the 1$^{st}$ and 2$^{nd}$ reach and gate, respectively.
7 CONCLUSION

The final canal model obtained is a multivariable linear model in the form of rational transfer functions of specified order that relate the increments of the gate positions with the increments of the reach downstream levels, with respect to the equilibrium defined by an average flow level.

In the approach followed, the model of the whole canal is obtained by combining basic blocks of canal stretches with gate models using constraints provided by flow equilibrium. This modular approach provides a flexible way to model canal networks with a given topology.

The fact that the model is based on basic physical principles allows to study the effect on the overall dynamics of the different parameters, such as the canal cross section geometry or the equilibrium flow.

This technique also allows to develop a pencil of models suitable for LPV controller design and to quantify uncertainty due to model approximation for robust controller design.

References


Figure 19: Block diagram of the three reaches of the experimental canal.
Figure 20: Block diagram of the three reaches, where $G_{ij}$ and $H_i$ are the transfer functions from the inputs to the outputs.