Training Sequence
Effect on Data-Aided Estimation of Synchronization Parameters

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Important notice

To the best of the authors knowledge, the results in this report are correct and accurate. However and due to its preliminary status, some errors may still subsist. Permission to use the results in this report is granted provided the results are duly acknowledged and referenced.
Abstract

In this report we examine the impact of training sequence design on the data-aided (DA) estimation of synchronization parameters. The analysis is done via a Cramér–Rao (CRB) approach. It is assumed that estimation is based on the observation of a linearly modulated information signal transmitted over an AWGN channel. A number of different, practically feasible and useful DA estimation scenarios is considered namely: i) estimation of the time delay $\tau$ with known carrier phase $\phi$ and known carrier frequency $\Omega$ ii) joint estimation of $\phi$ and $\Omega$ with $\tau$ known, iii) joint estimation of $\phi$ and $\tau$ with $\Omega$ known, iv) estimation of $\tau$ with $\Omega$ known and with $\phi$ irrelevant and v) joint estimation of $\phi$, $\Omega$ and $\tau$. It is found that time-symmetric (TS) or time-antisymmetric (TAS) sequences are particularly well-suited to DA synchronization parameter estimation in these contexts because their use leads to a diagonal Fisher information meaning there is no coupling between parameters to be estimated. In addition, it is also found that in these estimation contexts it is always possible to find a TS or TAS sequence for which the relevant CRB is minimum under a data sequence power constraint.

**Keywords:** Training sequence, data-aided estimation, synchronization, Cramér-Rao lower bound.
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1 Introduction

In this report we study the effect of the training sequence on the data-aided (DA) estimation of synchronization parameters. The analysis is done via a Cramér–Rao (CRB) approach. It is assumed that estimation is based on the observation of a linearly modulated information signal transmitted over an AWGN channel. This report is organized as follows. In Section 2 the signal model is introduced and the relevant Cramér–Rao (CRB) are derived. The effect of the data sequence on these CRBs is analyzed in Section 3 considering the following DA estimation contexts: i) estimation of $\tau$ with known $\phi$ and $\Omega$ (Section 3.1) ii) joint estimation of $\phi$ and $\Omega$ with $\tau$ known (Section 3.2), iii) joint estimation of $\phi$ and $\tau$ with $\Omega$ known (Section 3.3), iv) estimation of $\tau$ with $\Omega$ known and with $\phi$ irrelevant i.e., $\phi$ treated as a nuisance parameter (Section 3.4) and v) joint estimation of $\phi$, $\Omega$ and $\tau$ (Section 3.5). Finally, some conclusions are presented in Section 4.

Notation and definitions

1. Let the $K \times 1$ vector $\mathbf{x} = [x_0 \ x_1 \ \cdots \ x_{K-1}]^T$ represent an arbitrary complex-valued sequence. Then $\mathbf{x}_R$ denotes its time-reversed version i.e., $\{x_R(k) = x(K - 1 - k)\}_{k=0}^{K-1}$.

2. We say that a sequence $\{a_k\}_{k=0}^{K-1}$ of length $K$ symbols is time-symmetric (TS) or time-antisymmetric (TAS) if $\{a(k) = a(K - 1 - k)\}_{k=0}^{K-1}$ or $\{a(k) = -a(K - 1 - k)\}_{k=0}^{K-1}$ respectively. If a sequence $\{a_k\}_{k=0}^{K-1}$ is TAS and $K$ is odd, it is always assumed that $a_{K-1} = 0$.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Denotes</th>
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<tbody>
<tr>
<td>$(\cdot)^*$</td>
<td>scalar, vector or matrix conjugate</td>
</tr>
<tr>
<td>$(\cdot)^T$</td>
<td>vector or matrix transpose</td>
</tr>
<tr>
<td>$(\cdot)^H$</td>
<td>vector or matrix conjugate transpose</td>
</tr>
<tr>
<td>$i$</td>
<td>the imaginary operator $i = \sqrt{-1}$</td>
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2 Signal model and the Cramér–Rao lower bound

Assuming transmission over an AWGN channel, the complex baseband received signal may be represented as

$$
r(t) = \sqrt{E_s e^{i\phi} e^{2\pi f(t-t_0)}} \sum_{k=0}^{K-1} a_k p(t-kT-\tau) + w(t)$$

(1)
where $s(t | \phi, \Delta f, \tau, \{a_k\})$ denotes the signal component of $r(t)$; $E_s$ is the average symbol energy; $T$ is the symbol duration and $p(t)$ is a unit-energy, real-valued but otherwise arbitrary signaling pulse, possibly including the channel impulse response; $\tau$ is the time delay, $\phi$ is the carrier phase at the reference time $t_0$ and $\Delta f$ is the cyclic carrier frequency offset; in the sequel we consider the normalized angular frequency offset $\Omega \triangleq 2\pi \Delta f T$; $\{a_k\}_{k=0}^{K-1}$ is a sequence of $K$ known possibly complex data symbols which will be represented throughout by the vector $\mathbf{a} \triangleq [a_0 \ a_1 \ \cdots \ a_{K-1}]^T$; these symbols are from an arbitrary $M$-ary constellation $\mathcal{C}$ with zero-mean symbols $\{A_m\}_{m=0}^{M-1}$. $w(t)$ is a complex zero-mean Gaussian noise process with variance $\sigma_w^2$ and with independent real and imaginary components, each with variance $\sigma^2 = \sigma_w^2/2$ and bilateral power spectral density $N_0/2$. Without any loss of generality, we consider symbol sequences with normalized power $\mathbf{a}^H \mathbf{a} = K$. With this signal model the symbol signal-to-noise ratio is $E_s/N_0 = (\mathbf{a}^H \mathbf{a} / K) / \sigma_w^2 = 1/(2\sigma^2)$. Let $\mathbf{\theta} = [\theta_0 \ \theta_1 \ \theta_2]^T \triangleq [\phi \ \Omega \ \tau]^T$ denote the vector of synchronization parameters to be estimated and $\hat{\mathbf{\theta}} \triangleq [\hat{\phi} \ \hat{\Omega} \ \hat{\tau}]^T$ an unbiased estimate of $\mathbf{\theta}$ produced by the receiver based on $r(t)$. Then, the error covariance matrix for the joint estimation of the $\{\theta_i\}$ is lower bounded by the Cramér–Rao lower bound [10]

$$C_{\hat{\mathbf{\theta}}} = E_r [(\hat{\mathbf{\theta}} - \mathbf{\theta})(\hat{\mathbf{\theta}} - \mathbf{\theta})^T] \geq \mathbf{I}^{-1}(\mathbf{\theta}, \mathbf{a}) \triangleq \text{CRB}(\mathbf{\theta}) \quad (2)$$

where $\mathbf{I}(\mathbf{\theta}, \mathbf{a})$ is the Fisher information matrix (FIM) with entries [10]

$$I_{i,j}(\mathbf{\theta}, \mathbf{a}) = \frac{2E_s}{N_0} \int_{-\infty}^{\infty} \Re \left\{ \frac{\partial s(t | \mathbf{\theta}, \mathbf{a})}{\partial \theta_i} \frac{\partial s^*(t | \mathbf{\theta}, \mathbf{a})}{\partial \theta_j} \right\} dt. \quad (3)$$

Using (1) and (3) the Fisher matrix is computed as

$$\mathbf{I}(\mathbf{\theta}, \mathbf{a}) = \frac{2E_s}{N_0} \begin{bmatrix} Q_0 & Q_1 - \frac{t_0 - \tau}{T} Q_0 & \hat{Q}_0 \\ Q_1 - \frac{t_0 - \tau}{T} Q_0 & Q_2 - 2\frac{t_0 - \tau}{T} Q_1 + \left(\frac{t_0 - \tau}{T}\right)^2 Q_0 & \hat{Q}_1 - \frac{t_0 - \tau}{T} \hat{Q}_0 \\ \hat{Q}_0 & \hat{Q}_1 - \frac{t_0 - \tau}{T} \hat{Q}_0 & \hat{Q}_2 \end{bmatrix} \quad (4)$$

where $Q_m \triangleq \mathbf{a}^H \mathbf{F}_m \mathbf{a}$ and $\mathbf{F}_m$ is the generating matrix with entries

$$f_m(i, j) \triangleq \int_{-\infty}^{\infty} \left( \frac{t}{T} \right)^m p(t - iT)p(t - jT) dt \quad (5)$$

$\hat{Q}_m \triangleq \Im \{\mathbf{a}^H \hat{\mathbf{F}}_m \mathbf{a}\}$ and $\hat{\mathbf{F}}_m$ is the generating matrix with entries

$$\hat{f}_m(i, j) \triangleq \int_{-\infty}^{\infty} \left( \frac{t}{T} \right)^m \hat{p}(t - iT)p(t - jT) dt. \quad (6)$$

where $\hat{p}(t) \triangleq \frac{\partial \hat{p}(t)}{\partial t}$ and $Q_\tau \triangleq \mathbf{a}^H \hat{\mathbf{G}}_m \mathbf{a}$ where $\hat{\mathbf{G}}$ is the matrix with entries

$[\hat{\mathbf{G}}]_{i,j} = -\hat{g}(i + j)T]$, $0 \leq i, j \leq K - 1$ with $\hat{g}(t) \triangleq \frac{\partial \hat{g}(t)}{\partial t}$ and $g(t) \triangleq p(t) \otimes p(-t)$ is the receiver matched filter response to $p(t)$. Note that for any $p(t)$, $g(t) = g(-t)$ is even. Also,
\(f_0(i, j) = g[(i - j)T]\) and \(\hat{f}_0(i, j) = -\hat{g}[(i - j)T]\) where \(\dot{g}(t) \triangleq \frac{\partial g(t)}{\partial t} = -\dot{g}(-t)\) is odd. If \(z(t) \triangleq w(t) \otimes p(-t)\) denotes the matched filter response to \(w(t)\) then, as is easily shown, \(w^2 \hat{G}\) is the covariance matrix of the discrete-time noise process \(\{\dot{z}_k \triangleq \frac{\partial z(t)}{\partial t}\}_{t=kT}^{K-1}\); as such, \(\hat{G}\) is positive semidefinite [6, p. 190]. In addition, under the assumption that \(\hat{G}\) is nonsingular, it follows that \(\hat{G}\) is positive definite or, equivalently, that

\[
Q_\tau \triangleq a^H \hat{G} a > 0, \quad \forall a \neq 0.
\] (7)

For \(m\) even the matrix \(F_m\) is positive semi-definite. This is because \(\sigma_w^2 F_m\) is the covariance matrix of the discrete-time noise process \(z_k \triangleq \left((\frac{t}{T})^m/2 w(t)\right) \otimes p(-t)\}_{t=kT}. If we assume that \(F_m\) is nonsingular, then \(F_m\) is positive definite.

3 Effect of the training sequence

In this section we analyze the effect of the training sequence on the DA estimation of the time delay, carrier phase and frequency offset. This analysis is performed via a CRB approach. Ideally, the Fisher matrix (4) should be diagonal. In this case all parameters to be estimated would be uncoupled and could be estimated individually since no performance penalty would be incurred relative to the joint estimation of all parameters. If (4) is not diagonal then parameters are coupled and estimation is more difficult. The analytical representation of this effect is the fact that [8, p. 232]

\[
[I^{-1}(\theta, a)]_{i,i} \geq \frac{1}{I_i i(\theta, a)}
\] (8)

for any positive-definite Fisher information matrix \(I(\theta, a)\). This inequality states that if the parameters to be estimated are coupled then the ultimate performance achievable by any joint parameter estimator when estimating \(\theta_i\) is always inferior to that achievable when estimating \(\theta_i\) when all the others are known. This is true irrespective of the data sequence. Conversely, if the parameters are decoupled (the Fisher matrix is diagonal) then the performance of the joint estimator equals the performance of individual parameter estimator when all the other are known. This means that any training data sequence not rendering all parameters uncoupled will lead to a joint CRB(\(\theta_i\))joint which is larger than the minimum possible CRB(\(\theta_i\))individual (which is the CRB for the individual estimation of the parameter when all others are known). As a consequence of this we may conclude that optimum sequences are necessarily found among those that render parameters uncoupled and thus, to find an optimum sequence for the estimation of \(\theta_i\), if suffices to seek

\[
a_{opt} = \arg \min_{a \in \mathcal{S}} \{\text{CRB}(\theta_i)_{\text{individual}}\} = \arg \max_{a \in \mathcal{S}} \{I_{i,i}(\theta, a)\}
\] (9)
where $S$ denotes the set of all sequences for which the parameters are uncoupled. The formal proof of this important conclusion is given in the following theorem.

**Theorem 1.** Let $a_u$ be a data sequence for which parameters are uncoupled (i.e., $I(\theta, a_u)$ is diagonal) and $I_{i,i}(\theta, a_u)$ attains an absolute maximum $I_{i,i}(\theta, a)_{\text{max}}$ under the power constraint $a_u^H a_u = K$. Then $a_u$ is optimum in the sense that no other sequence exists that attains a smaller CRB under the same power constraint.

**Proof:** It suffices to consider data sequences for which parameters are coupled. The Cramér–Rao for the estimation of $\theta_i$ using $a_u$ is $\text{CRB}_u(\theta_i) = I^{-1}(\theta, a_u) = 1/I_{i,i}(\theta, a_u) = 1/I_{i,i}(\theta, a)_{\text{max}}$. Now let $a_c$ be a data sequence for which parameters are coupled. Then the CRB for the estimation of $\theta_i$ using $a_c$ is

$$
\text{CRB}_c(\theta_i) = I^{-1}(\theta, a_c) > \frac{1}{I_{i,i}(\theta, a_c)} \geq \frac{1}{I_{i,i}(\theta, a)_{\text{max}}} = \text{CRB}_u(\theta_i)
$$

where the first inequality is due to (8). This proves that no sequence $a_c$ can achieve a smaller CRB than $a_u$. Therefore, a sequence $a_{\text{opt}}$ that renders all parameters uncoupled (i.e., $a_{\text{opt}} \in S$) and for which $I_{i,i}(\theta, a_{\text{opt}})$ is maximum may not be unique but is optimum in the sense that no other sequence can achieve a lower CRB. In all synchronization contexts considered in this report, with exception for the joint estimation of all parameters examined in Section 3.5, the diagonal Fisher matrix entries are expressible as a single quadratic form on the data sequence i.e., $I_{i,i}(\theta, a) = a^H A a$ where $A$ is a positive-definite matrix with entries verifying the symmetry condition $a_{i,j} = a_{K-1-i,K-1-j}$. It is shown in Lemma 5 in the Appendix that if the dominant eigenvalue of $A$ is simple then the sequence $a$ which maximizes $I_{i,i}(\theta, a)$ under a power constraint $a^H a = K$ is necessarily TS or TAS. If it is not simple then non-TS or non-TAS sequences may exist that maximize $I_{i,i}(\theta, a)$. However, it is shown in Theorem A.3 that even in this case it is always possible to find a TS or TAS sequence for which $I_{i,i}(\theta, a)$ is maximum. In addition, in all these estimation contexts, it is shown in the Appendix that TS or TAS sequences render parameters uncoupled. Non-TS or non-TAS which attain the same performance (but no better) of TS or TAS sequences may exist but their determination is difficult because there is no systematic procedure to find them. On top of this, non-TS or non-TAS sequences do not in general render parameters uncoupled and are thus suboptimal when jointly estimating more than one parameter. For these reasons we will restrict attention to sequences that are either TS or TAS.

We now analyze each non-diagonal entry of (4) and study the impact of the data-sequence on the coupling between each pair of parameters:
1. Coupling between $\phi$ and $\tau$:

The coupling between $\phi$ and $\tau$ is represented by the entry $I_{0,2}(\theta, \mathbf{a}) = I_{2,0}(\theta, \mathbf{a}) = \mathbf{\dot{Q}}_0$. The matrix $\mathbf{\dot{F}}_0$ is neither positive-(semi)definite nor negative-(semi)definite. This is easy to see because $\mathbf{\dot{F}}_0$ is real and skew-symmetric [because $\dot{g}(t)$ is odd] and therefore its eigenvalues $\lambda_i$ are purely imaginary [1] (and complex conjugate). If $K$ is odd then one of the eigenvalues is zero and the other $K - 1$ are purely imaginary and complex conjugate. The eigenvalues of $\mathbf{\dot{F}}_0$ are thus $\epsilon_i = \pm \Im \{\lambda_i\}, i = 0, \ldots, \frac{K}{2} - 1$ if $K$ is even and $\epsilon_0 = 0$, $\epsilon_i = \pm \Im \{\lambda_i\}, i = 1, \ldots, \frac{K-1}{2}$ if $K$ is odd. This means that many sequences $\mathbf{a}$ may exist for which $\mathbf{\dot{Q}}_0 = 0$. However, it is shown in Theorem A.1 in the Appendix that if $\mathbf{a}$ is either a time-symmetric (TS) or time-antisymmetric (TAS) sequence then $\mathbf{\dot{Q}}_0 = 0$ no matter the pulse shape $p(t)$ used by the system. Also, if the sequence $\mathbf{a}$ is such that $\Re \{\mathbf{a}\} = \gamma \Im \{\mathbf{a}\}$ with $\gamma$ a real scalar, then it is proved in Theorem A.2 that $\mathbf{\dot{Q}}_0 = 0$ regardless of the pulse shape. This is an important characteristic of these sequences since they render the time delay and the carrier phase uncoupled.

2. Coupling between $\Omega$ and $\tau$:

The coupling between $\phi$ and $\tau$ is represented by the entry $I_{1,2}(\theta, \mathbf{a}) = I_{2,1}(\theta, \mathbf{a}) = \mathbf{\dot{Q}}_1 - \frac{t_0 - \tau}{T} \mathbf{\dot{Q}}_0$. In general, these entries are not zero. However, it is shown in Theorem A.2 in the Appendix that for large $K$ and i.i.d. data sequences (not necessarily TS or TAS) $I_{1,2}(\theta, \mathbf{a}) = I_{2,1}(\theta, \mathbf{a}) \approx 0$ revealing that in these conditions the time delay and carrier frequency are essentially uncoupled (this has been previously reported in [5]). However, if the sequence $\mathbf{a}$ is such that $\Re \{\mathbf{a}\} = \gamma \Im \{\mathbf{a}\}$ with $\gamma$ a real scalar, it is proved in Theorem A.2 that $I_{2,1}(\theta, \mathbf{a}) = I_{1,2}(\theta, \mathbf{a}) = 0$ and the time delay and carrier frequency are effectively decoupled regardless of the sequence length and sequence symbols (i.e., sequence symbols do not need to be i.i.d.). This is a strong argument in favor of the selection of such sequences for pilot-assisted joint estimation of $\tau$ and $\Omega$.

3. Coupling between $\phi$ and $\Omega$:

The coupling between $\phi$ and $\Omega$ is represented by the entry $I_{0,1}(\theta, \mathbf{a}) = I_{1,0}(\theta, \mathbf{a}) = Q_1 - \frac{t_0 - \tau}{T} Q_0$. It can be made zero by selecting the time instant $t_0$ such that $t_0 = \tau + \frac{TQ_1}{Q_0}$. This requires knowledge of $\tau$ and in general depends both on the data sequence and on the pulse shape $p(t)$. However, if the data sequence is either TS or TAS and the pulse shape $p(t)$ is either odd or even, then it is shown in Theorem A.4 in the Appendix that $t_0 = \tau + \frac{(K-1)T}{2}$ which still depends on the time delay but neither on the data sequence nor on the pulse shape.
From the preceding analysis we may conclude that the coupling between the time delay and the angular parameters (carrier phase and frequency) is or can be made zero or small. This justifies that i) $\tau$ be estimated assuming both $\phi$ and $\Omega$ are known and ii) $\phi$ and $\Omega$ be estimated jointly or independently (if $\phi$ and $\Omega$ can be made uncoupled) assuming $\tau$ is known. Both estimation scenarios are feasible in practice [3,4]. Additionally, two other feasible estimation contexts are considered: iii) joint estimation of the carrier phase and time delay with known carrier frequency offset and iv) estimation of the time delay with known frequency offset and with irrelevant carrier phase (i.e., $\phi$ treated as a nuisance parameter). The joint estimation of $\phi$, $\Omega$ and $\tau$ is also briefly considered. These estimation scenarios will be analyzed in the next Sections.

3.1 Data-Aided estimation of $\tau$ with known $\phi$ and $\Omega$

In this estimation context, both $\phi$ and $\Omega$ are known (or have been accurately estimated) and their effect removed from $r(t)$ and therefore we may set $\phi = 0$ and $\Omega = 0$ in (1). This estimation context is feasible because both $\phi$ and $\Omega$ may be accurately estimated without prior knowledge of $\tau$ [3]. From (2) we have $E_r[(\tilde{r} - \tau)^2] \geq \text{CRB}(\tau) = 1/I_{2,2}(\tau, a)$ and, from (4) the CRB is computed as

$$\text{CRB}(\tau) = \frac{1}{2(E_s/N_0)Q_\tau} = \frac{1}{2(E_s/N_0) a^H G a} \quad (11)$$

It is shown in Theorem A.3 in the Appendix that it is always possible to find a time-symmetric or time-antisymmetric sequence $a$ for which (11) is minimum under a power constraint $a^H a = K$.

3.2 Data-Aided joint estimation of $\phi$ and $\Omega$ with known $\tau$

In this Section we consider the joint estimation of the carrier phase $\phi$ and the normalized frequency offset $\Omega$, assuming the time delay $\tau$ is known or accurately estimated. This is a feasible estimation scenario because $\tau$ may be accurately estimated without prior knowledge of $\phi$, even in the presence of significant, unknown frequency offsets [3]. From (4), removing the line and column pertaining to $\tau$, the Fisher matrix becomes

$$I(\phi, \Omega, a) = \frac{2E_r}{N_0} \left[ \begin{array}{cc} Q_0 & Q_1 - \frac{\mu_\Omega}{\mu_\tau} Q_0 \\ Q_1 - \frac{\mu_\Omega}{\mu_\tau} Q_0 & Q_2 - 2\frac{\mu_\Omega}{\mu_\tau} Q_1 + \left(\frac{\mu_\Omega}{\mu_\tau}\right)^2 Q_0 \end{array} \right] \quad (12)$$

6
and the inverse Fisher matrix is given as

$$
\mathbf{I}^{-1}(\phi, \Omega, a) = \left( \frac{2E_s}{N_0} \left[ Q_0 \left( Q_2 - 2 \frac{t_0 - \tau}{T} Q_1 + \left( \frac{t_0 - \tau}{T} \right)^2 Q_0 \right) - \left( Q_1 - \frac{t_0 - \tau}{T} Q_0 \right)^2 \right] \right)^{-1} \\
\cdot \left[ Q_2 - 2 \frac{t_0 - \tau}{T} Q_1 + \left( \frac{t_0 - \tau}{T} \right)^2 Q_0 \right. \\
\left. - Q_1 + \frac{t_0 - \tau}{T} Q_0 \right].
$$

(13)

After simplification, the CRBs for the joint estimation of the carrier phase and frequency offset may be written

$$
\text{CRB}(\phi) = \left[ \mathbf{I}^{-1}(\phi, \Omega, a) \right]_{0,0} = \frac{1}{2E_s/N_0} \frac{Q_2 - 2 \frac{t_0 - \tau}{T} Q_1 + \left( \frac{t_0 - \tau}{T} \right)^2 Q_0}{Q_0 Q_2 - Q_1^2} 
$$

(14a)

and

$$
\text{CRB}(\Omega) = \left[ \mathbf{I}^{-1}(\phi, \Omega, a) \right]_{1,1} = \frac{1}{2E_s/N_0} \frac{Q_0}{Q_0 Q_2 - Q_1^2}.
$$

(14b)

As can be seen, the CRB(\Omega) does not depend on \(t_0 - \tau\). This is expected because the carrier phase is assumed unknown [7]. Since \(\tau\) is assumed known it is possible to decouple the carrier phase and the carrier frequency offset by adjusting \(t_0\) in order to verify \(I_{0,1}(\phi, \Omega, a) = I_{1,0}(\phi, \Omega, a) = 0\). From (12), this leads to

$$
\frac{t_0 = \tau + \frac{TQ_1}{Q_0}}{Q_0} 
$$

(15)

and, for this particular value of \(t_0\), the carrier phase CRB in (14a) attains its minimum [5, 7] which is

$$
\text{CRB}(\phi) = \frac{1}{2(E_s/N_0) Q_0}.
$$

(16)

Note that since now \(I(\phi, \Omega, a)\) is diagonal, \(\left[ \mathbf{I}^{-1}(\phi, \Omega, a) \right]_{i,i} = \left[ I_{i,i}(\phi, \Omega, a) \right]^{-1}, i = 0, 1\) and therefore the CRBs for the joint estimation of \(\phi\) and \(\Omega\) coincide with the CRBs for the individual estimation of either parameter when the other is known.

A nice property of either TS or TAS sequences is that for these \(\frac{Q_1}{Q_0} = \frac{K-1}{2}\), provided that \(p(t)\) is either odd or even\(^1\). This is shown in Theorem A.4 in the Appendix. Therefore, the value of \(t_0\) which decouples the estimation of \(\phi\) and \(\Omega\) becomes

$$
\frac{t_0 = \tau + \frac{(K - 1)T}{2}}{2}
$$

(17)

\(^1\)We will consider that \(p(t)\) is even since this is the case in most communication systems. Pulses \(p(t)\) with odd time-symmetry are of no interest for pulse-shaping purposes because \(p(0) = 0\).
which depends neither on the sequence nor on the pulse $p(t)$. From (14b) we may write

$$
\left\{ \frac{2E_s}{N_0} \text{CRB}(\Omega) \right\}^{-1} = Q_2 - \left( \frac{Q_1}{Q_0} \right)^2 Q_0 = Q_2 - \left( \frac{t_0 - \tau}{T} \right)^2 Q_0
$$

and

$$
= \mathbf{a}^H \left[ \int_{-\infty}^{\infty} \left( \frac{t}{T} - \frac{t_0 - \tau}{T} \right) p(t-iT) p(t-jT) dt \right] \mathbf{a} - 2 \mathbf{a}^H \left[ \int_{-\infty}^{\infty} \left( \frac{t_0 - \tau}{T} \right)^2 p(t-iT) p(t-jT) dt \right] \mathbf{a} - 2 \mathbf{a}^H \left[ \left( \frac{t_0 - \tau}{T} \right)^2 Q_0 - \left( \frac{t_0 - \tau}{T} \right) Q_1 \right] \mathbf{a} = \mathbf{a}^H \left[ \int_{-\infty}^{\infty} \left( \frac{t}{T} - \frac{t_0 - \tau}{T} \right)^2 p(t-iT) p(t-jT) dt \right] \mathbf{a} \quad (18)
$$

where the last equality follows because, from (15), $\frac{t_0 - \tau}{T} = \frac{Q_1}{Q_0}$. Therefore, (14b) may be written as

$$
\text{CRB}(\Omega) = \frac{1}{2\left(\frac{E_s}{N_0}\right) Q_\Omega} \quad (19)
$$

where $Q_\Omega \equiv \mathbf{a}^H \mathbf{F}'_2 \mathbf{a}$ and $\mathbf{F}'_2$ is the (positive-definite) generating matrix with entries

$$
f'_2(i,j) \triangleq \int_{-\infty}^{\infty} \left( \frac{t}{T} - \frac{t_0 - \tau}{T} \right)^2 p(t-iT) p(t-jT) dt \quad (20a)
$$

and

$$
= \int_{-\infty}^{\infty} \left( \frac{t}{T} - \frac{K-1}{2} \right)^2 p(t-iT) p(t-jT) dt. \quad (20b)
$$

Note that (20b) is only valid for TS or TAS sequences if $p(t)$ is odd or even (since in this case $\frac{t_0 - \tau}{T} = \frac{Q_1}{Q_0} = \frac{K-1}{2}$). According to the results in Theorem A.3 in the Appendix, it is always possible to find a TS or TAS sequence $\mathbf{a}$ for which either (16) or (19) is minimum under a power constraint $\mathbf{a}^H \mathbf{a} = K$.

### 3.3 Data-Aided joint estimation of $\phi$ and $\tau$ with known $\Omega$

In this Section we consider the joint estimation of the carrier phase $\phi$ and the time delay $\tau$, assuming the frequency offset $\Omega$ is known or accurately estimated. This is a feasible estimation scenario which may occur, for example, when a phase-locked loop (PLL) or other similar device is used to demodulate the received signal, producing a baseband information signal which remains
affected by an unknown phase rotation. From (4), removing the line and column pertaining to \( \Omega \), the Fisher matrix is

\[
I(\phi, \tau, a) = \begin{bmatrix} Q_0 & \dot{Q}_0 \\ \dot{Q}_0 & Q_\tau \end{bmatrix}.
\]

(21)

The inverse Fisher matrix is given as

\[
I^{-1}(\phi, \tau, a) = \frac{1}{2E_s/N_0} \begin{bmatrix} Q_\tau & -Q_0 \\ -Q_0 & Q_\tau - Q_0^2 \end{bmatrix}
\]

(22)

and the CRBs for the joint estimation of the carrier phase and time delay may thus be written

\[
\text{CRB}(\phi) = [I^{-1}(\phi, \tau, a)]_{0,0} = \frac{1}{2E_s/N_0} \frac{Q_\tau}{Q_0 Q_\tau - Q_0^2}
\]

(23a)

and

\[
\text{CRB}(\tau) = [I^{-1}(\phi, \tau, a)]_{1,1} = \frac{1}{2E_s/N_0} \frac{Q_0}{Q_0 Q_\tau - Q_0^2}
\]

(23b)

If \( \dot{Q}_0 = 0 \) then \( \phi \) and \( \tau \) are uncoupled and the CRBs in (23a) and (23b) reduce to the CRBs pertaining to the individual estimation of each parameter when all the others are known, i.e. (16) and (11) for \( \phi \) and \( \tau \) respectively. If the data sequence is either TS or TAS then, as stated earlier and demonstrated in Theorem A.1, \( \dot{Q}_0 = 0 \) regardless of the pulse shape used in the system. In this case, the CRBs in (23a) and (23b) are the inverse of a single quadratic form in the data sequence and, as shown in Theorem A.3, it is always possible to find a TS or TAS sequence \( a \) for which either CRB is minimized under a power constraint \( a^H a = K \). Therefore we conclude that in this estimation scenario the use of TS or TAS sequences is optimum since they not only render \( \phi \) and \( \tau \) uncoupled but also attain the minimum possible CRB.

### 3.4 Data-Aided estimation of \( \tau \) with \( \Omega \) known and \( \phi \) irrelevant

In this estimation scenario, after accurately estimating the carrier frequency offset and demodulating the received signal, the time delay is estimated independently of the (unknown) carrier phase. It is shown in [9] that for \( g(t) \) Nyquist, the relevant CRB is given by

\[
\text{CRB}(\tau) = \frac{1}{2E_s/N_0} \frac{Q_0}{F \left( K \frac{E_s}{N_0} \right) Q_0 Q_\tau - Q_0^2}
\]

(24)

where \( F \left( K \frac{E_s}{N_0} \right) \) is an integral function given by

\[
F(x) \triangleq 2x e^{-x} \int_0^\infty r e^{-xr^2} \frac{I_0^2(2xr)}{I_0(2xr)} \, dr
\]

(25)
and $I_n(z)$ is the modified Bessel function of the first kind and order $n$. Comparing (24) with (23b) we see that they only differ by the factor $F\left(\frac{K E_s}{N_0}\right)$ which does not depend on the data sequence but affects only the equivalent SNR. In [9] it is shown that $0 < F\left(\frac{K E_s}{N_0}\right) < 1$ and thus, the CRB in (24) is always greater than the CRB in (23b) meaning that the estimation of $\tau$ considering $\phi$ irrelevant is always more difficult than the estimation of $\tau$ jointly with $\phi$. The similarity between these two CRBs and the fact that $F\left(\frac{K E_s}{N_0}\right)$ does not depend on the data sequence allows us to conclude that the use of TS or TAS sequences is also optimal in this estimation context. However, this is valid only when $g(t)$ is Nyquist. When this is not the case, the function $F\left(\frac{K E_s}{N_0}\right)$ in (24) becomes $F\left(Q_0 \frac{E_s}{N_0}\right)$ which depends on the data sequence through the quadratic $Q_0$. However in this case the dependence is weak, because $Q_0 \approx K$ particularly for large $K$ and even for relatively low values of the normalized pulse bandwidth.

There is another route to overcome the difficulty posed by the fact that $Q_0$ depends on the data sequence, which consists on minimizing the CRB subject to the constraint $Q_0 = a^H F_0 a = K$, instead of the usual constraint $a^H a = K$. It is easy to show that in this case, the maximization of the quadratic $a^H A a$ (with $A$ a general matrix) subject to $Q_0 = K$ is achieved when $a$ is the dominant eigenvector of $F_0^{-1} A$ and equals its largest eigenvalue. This method allows optimum sequences to be found for an arbitrary pulse shape $p(t)$. Note that the resulting sequence may require additional renormalization if the condition $a^H a = K$ is to be satisfied.

### 3.5 Data-Aided joint estimation of $\phi$, $\Omega$ and $\tau$

In this case we must consider the complete Fisher matrix in (4). However, it is shown in Theorem A.1 in the Appendix that for TS or TAS sequences, $I_{0,2}(\theta, a) = I_{2,0}(\theta, a) = 0$ regardless of $t_0$ and pulse shape $p(t)$. This means that $\tau$ and $\phi$ are uncoupled and therefore, with these sequences, knowing $\tau$ is immaterial for the estimation of $\phi$. In addition, it is shown in Theorem A.2 in the Appendix that $I_{1,2}(\theta, a) = I_{2,1}(\theta, a)$ does not depend on either $t_0$ or $\tau$ and becomes increasingly small as the sequence length $K$ increases, provided the symbols are independent and identically distributed (i.i.d.). This means that the time delay $\tau$ and the frequency offset $\Omega$ are essentially uncoupled and thus, not knowing $\tau$ should not significantly perturb the estimation of $\Omega$.

### 4 Conclusions

In this report we have presented and proved some results pertaining to the impact of the training sequence on the data-aided estimation of synchronization parameters. Special attention has been given to the use of time-symmetric or time-antisymmetric data sequences. In particular,
it is found that for these sequences the coupling between the time delay $\tau$ and the carrier phase $\phi$ is zero. A number of different estimation context has been examined namely: i) estimation of $\tau$ with known $\phi$ and carrier frequency $\Omega$ ii) joint estimation of $\phi$ and $\Omega$ with $\tau$ known, iii) joint estimation of $\phi$ and $\tau$ with $\Omega$ known, iv) estimation of $\tau$ with $\Omega$ known and with $\phi$ irrelevant and v) joint estimation of $\phi$, $\Omega$ and $\tau$. It is also shown that in contexts i) through iv) it is always possible to find a TS or TAS data sequence $\mathbf{a}$ which minimizes the relevant CRB under a power constraint $\mathbf{a}^H\mathbf{a} = K$. 


Appendix

In this Appendix we state and prove some results pertaining to the effect of the training sequence on the DA estimation of synchronization parameters. These results are presented in Theorems A.1 through A.4 below. A number of useful, ancillary results are first presented and proved in the following lemmas.

Lemma 1. Let $A$ be a real symmetric (skew-symmetric) matrix. Then the scalar quadratic $Q \triangleq x^H A x$ is purely real (imaginary).

Proof: If $A$ is real and symmetric, $A = A^H$. Then $Q^* = Q^H = (x^H A x)^H = x^H A^H x = x^H A x = Q$ and $Q$ is purely real. If $A$ is real and skew-symmetric, $A = -A^H$. Then $Q^* = Q^H = (x^H A x)^H = x^H A^H x = -x^H A x = -Q$ and $Q$ must be purely imaginary. \hfill \Box

Lemma 2. Let $A$ be a matrix satisfying either $a_{i,j} = a_{K-1-i,K-1-j}$ or $a_{i,j} = -a_{K-1-i,K-1-j}$. Then $x_R^H A x_R = x^H A x$ or $x_R^H A x_R = -x^H A x$ respectively.

Proof: Let $P$ denote the real, symmetric, permutation matrix with entries $P_{i,j} = \delta(K-1-i-j)$ (note that $P^T = P$ and $PP = I$). Then $x_R = Px$ and $x_R^H A x_R = (Px)^H A (Px) = x^H (PAP)x$. Left(right)-multiplying $A$ by $P$ causes the lines (columns) of $A$ to commute i.e., line (column) $i$ becomes line (column) $K-1-i$ and therefore $PAP$ has entries $a_{K-1-i,K-1-j}$ which equal $a_{i,j}$ or $-a_{i,j}$ by assumption. As a result $PAP = A$ and $x_R^H A x_R = x^H A x$ or $PAP = -A$ and $x_R^H A x_R = -x^H A x$. \hfill \Box

Lemma 3. Let $A$ be a symmetric matrix satisfying $a_{i,j} = a_{K-1-i,K-1-j}$ and $P$ a permutation matrix as defined in the proof of Lemma 2. Then $A$ and $P$ commute i.e., $PA = AP$.

Proof: First note that $a_{i,K-1-j} = a_{K-1-i,j}$ meaning that $A$ is symmetric. Let $Q = AP$ with entries $q_{i,j}$. Because of the right multiplication by $P$, which commutes the columns of $A$, we have $q_{i,j} = a_{i,K-1-j}$ and therefore $q_{K-1-i,K-1-j} = a_{K-1-i,j} = a_{i,K-1-j}$ which the last equality is justified because $A$ is symmetric. From this we conclude that $q_{i,j} = q_{K-1-i,K-1-j}$ and, since $PP = I, PA = P(AP)P = PQP$, it follows from the proof of Lemma 2 that $PA = PQP = Q$ or, equivalently $PA = AP$. \hfill \Box

Lemma 4. Let $A$ be a real matrix satisfying $a_{i,j} = a_{K-1-i,K-1-j}$ and $x_{opt}$ the vector which maximizes the quadratic $x^H A x$ subject to $\|x\| = 1$ i.e., $x_{opt}$ is the eigenvector of $A$ corresponding to its dominant eigenvalue $\lambda_D$, and $x_{opt}^H A x_{opt} = \lambda_D$. Then the time symmetric sequences $y_{opt} = \pm \frac{x_{opt} + x_{opt,R}}{\|x_{opt} + x_{opt,R}\|}$ and the time anti-symmetric sequences $y_{opt} = \pm \frac{x_{opt} - x_{opt,R}}{\|x_{opt} - x_{opt,R}\|}$ are such that $\|y_{opt}\| = 1$ and $y_{opt}^H A y_{opt} = \lambda_D$. 

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Proof: The proof that $\|y_{\text{opt}}\| = 1$ is trivial. Now let $z$ be an arbitrary complex sequence and $y = \frac{z \pm z_R}{\|z \pm z_R\|}$. Then
\[
y^H Ay = \left( \frac{z \pm z_R}{\|z \pm z_R\|} \right)^H A \left( \frac{z \pm z_R}{\|z \pm z_R\|} \right) \tag{A.1a}
\]
\[
= \frac{(z \pm z_R)^H A (z \pm z_R)}{(z \pm z_R)^H (z \pm z_R)} \tag{A.1b}
\]
\[
= \frac{z^H Az \pm z^H Az_R \pm z_R^H Az + z_R^H Az_R}{z^H z \pm z^H z_R \pm z_R^H z + z_R^H z_R} \tag{a}
\]
\[
= \frac{z^H (I \pm P) Az \pm z^H (AP)z \pm z^H (PA)z + z^H (PAP)z}{z^H z \pm z^H z_R \pm z_R^H z + z_R^H z_R} \tag{b}
\]
where equality (a) follows because $P^H = P$ and equality (b) because $PP = I$, $PAP = A$ (proof of Lemma 2) and $PA = AP$ (Lemma 3). To proceed, let $R = \frac{z^H Bz}{z^H Cz}$ with $B$ and $C$ arbitrary matrices and consider the maximization problem $z_{\text{opt}} = \max_z \{ R \}$ subject to $z^H Cz = 1$. Forming the Lagrangean $L(z, \lambda) = z^H Bz - \lambda (z^H Cz - 1)$ we may conclude that $z_{\text{opt}}$ is the eigenvector corresponding to the dominant eigenvalue of $C^{-1} B$ and that $R_{\text{max}} = \frac{z_{\text{opt}}^H Bz_{\text{opt}}}{z_{\text{opt}}^H Cz_{\text{opt}}}$ equals this eigenvalue. From (A.1b) we have $B = (I \pm P)A$ and $C = I \pm P$ so $C^{-1} B = A$ even if $C$ is singular, meaning that $z_{\text{opt}}$ and $x_{\text{opt}}$ must be the same and $R_{\text{max}} = \lambda_D$. The result for $y_{\text{opt}}$ in the Lemma follows immediately after setting $z = x_{\text{opt}}$ in (A.1a).

\[\square\]

Lemma 5. Let $A$ be a real positive-definite matrix satisfying the condition $a_{i,j} = a_{K-1-i,K-1-j}$ and define the quadratic $Q \triangleq x^H Ax$. Then, it is always possible to find a time-symmetric or time-antisymmetric sequence $x_{\text{opt}}$ which maximizes $Q$ subject to $\|x_{\text{opt}}\| = 1$.

Proof: If the dominant eigenvector of $A$ is simple (unique), then the only sequences maximizing $Q$ are $\pm x_{\text{opt}}$. But from Lemma 2, the sequences $\pm x_{\text{opt},R}$ also maximize $Q$. Therefore $x_{\text{opt}}$ must equal $\pm x_{\text{opt},R}$ and thus $x_{\text{opt}}$ is necessarily time-symmetric or time-antisymmetric. If the dominant eigenvector is not simple, then several solutions $x_{\text{opt}}$ (eigenvectors of $A$), not necessarily time-symmetric or time-antisymmetric, may exist. However, from Lemma 4, using one (any) of the $x_{\text{opt}}$ it is always possible to find a sequence $y_{\text{opt}}$ which is time-symmetric or time antisymmetric and for which $y_{\text{opt}}^H Ay_{\text{opt}} = x_{\text{opt}}^H Ax_{\text{opt}}$.

\[\square\]

We now state a number of properties regarding the generating matrices $\hat{G}$, $F_0$ $F_1$ and $F_2'$, which are used in the proof of theorems A.3 and A.4.
P1. Since $[\tilde{G}]_{i,j} = \tilde{g}(t)|_{t=(i-j)T}$ and $\tilde{g}(t)$ is even, $[\tilde{G}]_{k-1-i,k-1-j} = [\tilde{G}]_{i,j}$ for any pulse $p(t)$.

P2. Since $[\tilde{F}_0]_{i,j} = g(t)|_{t=(i-j)T}$ and $g(t)$ is even, $[\tilde{F}_0]_{k-1-i,k-1-j} = [\tilde{F}_0]_{i,j}$ for any pulse $p(t)$.

P3. If $p(t)$ is either odd or even, the entries of the matrix $\tilde{F}_0$ satisfy $f_0(K-1-i, K-1-j) = f_0(i, j)$. This is easily proved using (5).

P4. If $p(t)$ is either odd or even, the entries of the matrix $\tilde{F}_2'$ satisfy $f_2'(K-1-i, K-1-j) = f_2'(i, j)$. This is easily proved using (20b).

P5. Using standard Fourier techniques it is possible to show that if $p(t)$ is either odd or even, then

$$\hat{f}_0(i, j) = \hat{g}[(i-j)T],$$

$$f_1(i, j) = \frac{i+j}{2} g[(i-j)T]$$

and

$$\hat{f}_1(i, j) = \frac{i+j}{2} \hat{g}[(i-j)T] - \frac{1}{2T} g[(i-j)T].$$

Theorem A.1. If the sequence $\mathbf{a}$ is either TS or TAS then the Fisher matrix entries $I_{0,2}(\theta, \mathbf{a})$ and $I_{2,0}(\theta, \mathbf{a}) = \hat{Q}_0$ given by (4) are identically zero regardless of the pulse shape $p(t)$.

Proof: From (6) we have that $\hat{f}_0(i, j) = -\hat{g}[(i-j)T]$ which is odd meaning that $\hat{\tilde{F}}_0$ is skew-symmetric and Toeplitz. Therefore $\hat{f}_0(i, j) = -\hat{f}_0(K-1-i, K-1-j)$. Let $P$ be the permutation matrix as defined in the proof of Lemma 2. If the sequence $\mathbf{a}$ is TS or TAS then $\mathbf{a} = \frac{a+ar}{2}$ and, recalling that $\mathbf{a}_R = P\mathbf{a}$

$$\hat{Q}_0 = \mathbf{a}^H \hat{\tilde{F}}_0 \mathbf{a} = \left(\frac{a+ar}{2}\right)^H \hat{\tilde{F}}_0 \left(\frac{a+ar}{2}\right)$$

$$= \mathbf{a}^H \left(\frac{I+P}{2}\right)^H \hat{\tilde{F}}_0 \left(\frac{I+P}{2}\right) \mathbf{a}$$

$$= \frac{1}{4} \mathbf{a}^H \left(\hat{\tilde{F}}_0 + P\hat{\tilde{F}}_0 P + \hat{\tilde{F}}_0 P + P\hat{\tilde{F}}_0\right) \mathbf{a}$$

From the properties of $P$ and $\hat{\tilde{F}}_0$ it follows that $P\hat{\tilde{F}}_0 P = -\hat{\tilde{F}}_0$. In addition

$$\hat{\tilde{F}}_0 P + P\hat{\tilde{F}}_0 = P^{-1}(\hat{\tilde{F}}_0 P + PP\hat{\tilde{F}}_0 F_0) = 0$$

and from (A.3) we conclude that $\hat{Q}_0 = 0$. \qed
Since $\hat{Q}_0$ is the FIM entry relating the carrier phase $\phi$ and the time delay $\tau$ this result means that these parameters are uncoupled.

**Theorem A.2.** If the sequence $a$ is i.i.d. then the Fisher matrix entries $I_{1,2}(\theta, a)$ and $I_{2,1}(\theta, a) = I_{1,2}(\theta, a) = \hat{Q}_1 - t_0 - \tau \hat{Q}_0$ given in (4) i) do not depend on either $t_0$ or $\tau$ and ii) become increasingly small as the sequence length $K$ increases. In addition, iii) if the sequence is either TS or TAS then these entries do not depend on either $t_0$ or $\tau$ and iv) if the sequence is such that $\Re\{a\} = \Re\{a\}$ with $\gamma$ a real scalar, then $I_{2,1}(\theta, a)$ and $I_{1,2}(\theta, a)$ are identically zero.

**Proof:** To prove parts i) and ii) we consider that the data sequence is i.i.d., ergodic and long ($K$ is large) and approximate the Fisher matrix entry by its mean over the data sequence $a$ [2,5].

This yields

$$E_a \left[ \hat{Q}_1 - \frac{t_0 - \tau}{T} \hat{Q}_0 \right] = E_a \left[ \Re \left\{ a^H \hat{F}_1 a - \frac{t_0 - \tau}{T} a^H \hat{F}_0 a \right\} \right]$$

$$= \Re \left\{ E_a \left[ a^H \hat{F}_1 a - \frac{t_0 - \tau}{T} a^H \hat{F}_0 a \right] \right\}$$

$$= \Re \left\{ \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} \hat{f}_1(i, j) E_a[a_i a_j^*] - \frac{t_0 - \tau}{T} \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} \hat{f}_0(i, j) E_a[a_i a_j^*] \right\}$$

$$\approx \Re \left\{ \sum_{i=0}^{K-1} \hat{f}_1(i, i) E_a[|a_i|^2] - \frac{t_0 - \tau}{T} \sum_{i=0}^{K-1} \hat{f}_0(i, i) E_a[|a_i|^2] \right\}$$

(A.5a)

$$= 0$$

(A.5b)

where (A.5a) is valid for i.i.d. symbols and for large $K$ and (A.5b) because both $\hat{f}_0(i, j)$ and $\hat{f}_1(i, j)$ are real [c.f. (6)] and in addition $\hat{f}_0(i, i) = -\dot{g}(0) = 0$. From theorem A.1, $\hat{Q}_0 = 0$ for TS or TAS sequences. Therefore $I_{2,1}(\theta, a) = I_{1,2}(\theta, a) = \hat{Q}_1$ which does not depend on either $t_0$ or $\tau$ proving claim iii). Theorem A.5 asserts that for data sequences as in claim iv), $\hat{Q}_1 = \hat{Q}_0 = 0$. As a consequence, $I_{2,1}(\theta, a) = I_{1,2}(\theta, a) = 0$.

**Theorem A.3.** It is always possible to find a TS or TAS sequence for which either $Q_\tau$, $Q_0$ or $Q_\Omega$ are maximum.

**Proof:** Since $Q_\tau$, $Q_0$ and $Q_\Omega$ are positive-definite quadratic forms and their generating matrices $\hat{G}$, $\hat{F}_0$ and $\hat{F}_0'$, respectively, satisfy the condition on Lemma 5 (properties P1-P4 above) it follows from this Lemma that it is always possible to find a time-symmetric or time-antisymmetric sequence which maximizes either $Q_\tau$, $Q_0$ or $Q_\Omega$. \hfill $\Box$
Theorem A.4. Let $Q_m \triangleq a^H F_m a = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} a_i a_j^* f_m(i,j)$ where $F_m$ is the matrix with entries $f_m(i,j)$ as in (5) with $p(t)$ either odd or even and $a$ is a time-symmetric or time-antisymmetric sequence. Then $Q_0 = \frac{K-1}{2}$.

Proof: If $K$ is even $\{a_i\}_{i=0}^{\frac{K}{2}-1} = \pm\{a_{K-1-i}\}_{i=0}^{\frac{K}{2}-1}$. Then, with $g_{i,j} = g[(i-j)T]$

$$Q_0 = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} a_i a_j^* f_0(i,j) = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} a_i a_j^* g_{i,j}$$

$$= \sum_{i=0}^{\frac{K}{2}-1} \sum_{j=0}^{\frac{K}{2}-1} \left( a_i [a_j^* g_{i,j} + a_{K-1-j}^* g_{i,K-1-j}] + a_{K-1-i} [a_j^* g_{K-1-i,j} + a_{K-1-j}^* g_{K-1-i,K-1-j}] \right)_{\pm a_j}$$

$$= 2 \sum_{i=0}^{\frac{K}{2}-1} \sum_{j=0}^{\frac{K}{2}-1} a_i a_j^* (g_{i,j} \pm g_{i,K-1-j}) \quad (A.6)$$

and, using (A.2b)

$$2Q_1 = 2 \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} a_i a_j^* f_1(i,j) = \sum_{i=0}^{K-1} \sum_{i=0}^{K-1} a_i a_j^* (i+j) g_{i,j}$$

$$= \sum_{i=0}^{\frac{K}{2}-1} \sum_{j=0}^{\frac{K}{2}-1} \left( a_i [a_j^* (i+j) g_{i,j} + a_{K-1-j}^* (i+K-1-j) g_{i,K-1-j}] \right)_{\pm a_j}$$

$$+ a_{K-1-j} [a_j^* (K-1-i+j) g_{K-1-i,j} + a_{K-1-j}^* (K-1-i+K-1-j) g_{K-1-i,K-1-j}] \right)_{\pm a_j}$$

$$= 2(K-1) \sum_{i=0}^{\frac{K}{2}-1} \sum_{j=0}^{\frac{K}{2}-1} a_i a_j^* (g_{i,j} \pm g_{i,K-1-j}) \quad (A.7)$$

The result $\frac{Q_1}{Q_0} = \frac{K-1}{2}$ follows immediately from the comparison of (A.6) with (A.7). If $K$ is odd $\{a_i\}_{i=0}^{\frac{K-1}{2}} = \pm\{a_{K-1-i}\}_{i=0}^{\frac{K-1}{2}}$, $a_{\frac{K-1}{2}}$ is arbitrary if the sequence is time-symmetric and $a_{\frac{K-1}{2}} = 0$ if the sequence is time-antisymmetric. Following the same steps as previously for $K$ even, it may be shown that

$$Q_0 = \begin{cases} 
2 \sum_{i=0}^{K-3} \sum_{j=0}^{K-3} a_i a_j^* (g_{i,j} - g_{i,K-1-j}), & \text{a is TAS} \\
2 \sum_{i=0}^{K-3} \sum_{j=0}^{K-3} a_i a_j^* (g_{i,j} + g_{i,K-1-j}) + 4 \sum_{i=0}^{K-3} \Re \left\{ a_i a_{K-1}^* \right\} g_{i,\frac{K-1}{2}} + \left| a_{\frac{K-1}{2}} \right|^2 g_{\frac{K-1}{2},\frac{K+1}{2}}, & \text{a is TS} 
\end{cases}$$

and that, for $a$ either time-symmetric or time-antisymmetric, $2Q_1 = (K-1) Q_0$.  

□
Theorem A.5. Let \( \hat{Q}_m \triangleq \Re\{a^H \hat{F}_m a\} = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} \Re\{a_i a_j^*\} \hat{f}_m(i, j) \) where \( \hat{F}_m \) is the matrix with (real) entries \( \hat{f}_m(i, j) \) as in (6). Then for data sequences \( a = y + j\gamma y \) where \( y \) is a real or a purely imaginary sequence and \( \gamma \) an arbitrary (possibly complex) scalar, \( \hat{Q}_m = 0 \).

Proof: From the definition we have, noting that \( a = (1 + j\gamma) y \)

\[
\hat{Q}_m \triangleq \Re\{a^H \hat{F}_m a\} = \Re\{|1 + j\gamma|^2 y^H \hat{F}_m y\} = 0
\]  \hspace{1cm} (A.8)

because, for \( y \) real or purely imaginary the quadratic \( y^H \hat{F}_m y \) is real.
References


