MODEL REFERENCE ADAPTIVE CONTROL OF ELECTRODYNAMIC LOUDSPEAKERS

Ricardo Adriano Ribeiro ∗ Gonçalo Gomes Tavares ∗∗ João Miranda Lemos ∗∗∗

∗ INESC-ID Rua Alves Redol, 9 1000-029 Lisboa Portugal (e-mail: ricardo.a.ribeiro@inesc-id.pt)
∗∗ INESC-ID (e-mail: goncalo.tavares@inesc-id.pt)
∗∗∗ INESC-ID (e-mail: jlml@inesc-id.pt)

Control; Adaptive; Nonlinear; Distortion; Loudspeaker; Parameter estimation

Abstract: Electrodynamic loudspeakers exhibit nonlinear behavior often distorting the acoustic signal they try to reproduce. This report will describe controllers which are able to effectively reduce the effects of the nonlinearities of the loudspeaker. Those controllers will be based on the feedback linearization technique. In order to compensate the frequency response of the loudspeaker we will use pole placement. In addition, the controller will be made adaptive by application of the model reference adaptive control topology to the feedback linearization controller with pole placement.

An electrodynamic loudspeaker is an electroacoustic transducer that exhibits nonlinear behavior (Klippel [1989], Adriano Ribeiro [2005], Ribeiro et al. [2005], Ribeiro [2004]). This means that it often distorts the acoustic signal it tries to reproduce, specially if one desires to reproduce low frequency signals with high amplitude. In this situation, the loudspeaker cone is forced to have greater displacements which will cause larger distortions since the loudspeaker nonlinearities depend essentially on the cone displacement.

In this work, controllers which are able to effectively reduce the effects of the nonlinearities of the loudspeaker will be determined. Those controllers will be based on the feedback linearization technique and, in order to compensate the frequency response of the loudspeaker we will use pole placement. In addition, the controller will be made adaptive by application of the model reference adaptive control topology to the feedback linearization controller with pole placement.

1. THE LOUDSPEAKER

The loudspeaker can be represented by a nonlinear state space model

\[
\dot{x} = f(x) + g(x)u
\]

where

\[
f(x) = \begin{bmatrix}
-R \frac{x_2}{m} - \frac{k(x_1)}{m} x_1 + \frac{Bl(x_1)}{m} x_3 + \frac{L_x(x_1)}{2m} x_3^2 \\
-\frac{Bl(x_1)}{L(x_1)} x_2 - \frac{L_x(x_1)}{L(x_1)} x_2 x_3 - \frac{R_e}{L(x_1)} x_3
\end{bmatrix}
\]

and

\[
g(x) = \begin{bmatrix} 0 & 0 & \frac{1}{L(x_1)} \end{bmatrix}^T.
\]

A special notation is used to represent the derivatives of the inductance \( L(x) \) with respect to the displacement \( x \): \( L_x(x) = \frac{\partial}{\partial x} L(x) \) and \( L_{xx}(x) = \frac{\partial^2}{\partial x^2} L(x) \).

The state vector is \( x = [x_1 \ x_2 \ x_3]^T = [x \ v \ i]^T \) and the variables
\[ B_l(x) = b_l_0 + b_l_1 x + b_l_2 x^2 + \cdots + b_l_n x^n \]  
\[ L(x) = l_0 + l_1 x + l_2 x^2 + \cdots + l_n x^n \]  
\[ k(x) = k_0 + k_1 x + k_2 x^2 + \cdots + k_n x^n \]  
represent the polynomial approximations of the unknown nonlinear functions representing the variations of the corresponding loudspeaker parameters \((B_l, L \text{ and } k)\) with the displacement \(x\) of the loudspeaker cone. For the output equation we may have the vector function

\[
y = h(x) = \left[ x \ V \ a \ \dot{i} \right]^T = 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
-\frac{R}{m^2} - \frac{k(x_1)}{m} x_1 + \frac{B_l(x_1)}{m} x_3 + \frac{L(x_1)}{2m} x_3^2 
\end{bmatrix}
\]

or a subset of rows from it according to the signals one has an interest at the output.

## 2. FEEDBACK LINEARIZATION

If we differentiate the output \(y\) \(\gamma\) times we obtain the expression

\[
y^{(\gamma)} = L_{gT}^\gamma h(x) + L_{g} L_{lT}^{\gamma-1} h(x) u \tag{7}
\]

The operator \(L_{lT} h(x)\) represents the Lie derivative of \(h(x)\) with respect to \(f(x)\) an is given by \(L_{lT} h(x) = \frac{\partial h(x)}{\partial x} f(x)\). The expression \(L_{g} L_{lT} h(x)\) represents the application of the lie derivative operator twice, each time with respect to a different function and \(L_{gT}^\gamma h(x)\) represents the application of the Lie derivative operator \(\gamma\) times with respect to the same function. The relative degree \(\gamma\) is the smallest integer for which \(L_{g} L_{lT}^{\gamma-1} h(x) \neq 0\).

A control signal \(u\) able to eliminate all the nonlinear functions is given by

\[
u = \frac{1}{L_{g} L_{lT}^{\gamma-1} h(x)} (-L_{lT}^\gamma h(x) + q) \tag{8}
\]

and after applying it to equation (7) we end up with system who’s input to output relation is a chain of \(\gamma\) integrators

\[
y^{(\gamma)} = q. \tag{9}
\]

thus linearizing the system. This process just described is what is named as Feedback Linearization.

Selecting the position \(x = x_1\) as the loudspeaker output \((h(x) = x_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x)\) we determine that the relative degree of the loudspeaker is \(\gamma = 3\). Since the order of the state space equations is also 3, then there are no zero-dynamics (Khalil [2002]). The acceleration \(a\), representing the sound a person can hear from the speaker, is the second time derivative of the position \(x\). Consequently, it is also linearized.

The Lie derivatives required for the control law equation (8) are

\[
L_{g} L_{lT}^2 h(x) = \frac{x_3 \left( \frac{d^2}{dx_1^2} L(x_1) \right)}{m L(x_1)} + \frac{B_l(x_1)}{m L(x_1)} \tag{10}
\]

and

\[
L_{g} L_{lT}^3 h(x) = 
\begin{align*}
&\frac{x_2 R^2}{m^2} - \frac{x_2^2 R^2}{m^2} \left( \frac{d}{dx_1} L(x_1) \right) R - \frac{B_l(x_1) x_1 R}{m^2} + \frac{k(x_1) x_1 R}{m^2} + \frac{L(x_1)}{2m} x_3^2 - \frac{R e x_3^2 \left( \frac{d}{dx_1} L(x_1) \right)}{m L(x_1)} \\
&\quad \vdots 
\end{align*}
\tag{11}
\]

(NOTE: the previous equation is too large, only a small part is shown) We can see that the controller is already quite complex. However, some complexity is still hidden: the polynomials for \(L(x), B_l(x)\) and \(k(x)\) (equations (3), (4) and (5)) are still missing and their introduction will significantly increase the complexity of the expressions.

It will be necessary to make this controller an adaptive one but such a task will be impossible to achieve using the complex expressions found in equations (10) and (11). Instead, let us try to describe those Lie derivatives in a linear form with respect to the parameters: \(L_{lT} h(x) = \theta^T w^1\) and \(L_{g} L_{lT} h(x) = \theta^T w^2\), where \(\theta^1\) and \(\theta^2\) are vectors with the parameters to be adapted and \(w^1\) and \(w^2\) are vectors with known signals, independent from any parameter, that can be computed from the state variables and the outputs of the loudspeaker.

Looking at equations (10) and (11) we see that it is impossible to get such a description due to the divisions by \(L(x_1)\). However, this can be overcome by defining an additional approximated polynomial

\[
L_{inv}(x_1) = l_{inv_0} + l_{inv_1} x_1 + l_{inv_2} x_1^2 + \cdots + l_{inv_n} x_1^n \tag{12}
\]

such that \(L_{inv}(x_1) \approx \frac{1}{L(x_1)}\). Now we replace all the divisions with \(L(x_1)\) by multiplications with \(L_{inv}(x_1)\). Note that, in general, \(l_{inv} \neq \frac{1}{n}\) and that most parameters \(l_n\) will still appear due to the expressions where \(L(x)\) appears in the numerator. Only the \(L(x)\) polynomials that appear in the denominator are replaced. So, after some extensive calculations, we end up with the following expressions:
for our final system. We can do this by applying the following pole placement law:

\[
q = -p_0x - p_1v - p_2a + p_3r = -p_0z_1 - p_1z_2 - p_2z_3 + p_3r
\]

(16) in equation (9). Then we get the following linear state space system

\[
\dot{z} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\rho_0 & -\rho_1 & -\rho_2 & \rho_3
\end{bmatrix} z + \begin{bmatrix}
0 \\
0 \\
r
\end{bmatrix} r
\]

(17) which has as a transfer function

\[
\frac{Z_1}{R} = \frac{p_3}{s^3 + p_2s^2 + p_1s + p_0}
\]

(18) Recall that \( z = \begin{bmatrix} x \ v \ a \end{bmatrix}^T \), so \( Z_1 \) is the Laplace transform of the position \( x \). Then the above expression gives the transfer function from the control input \( r \) to the loudspeaker cone position \( x \). The expression is similar to the transfer function for the linear approximation of the loudspeaker, it is a third order low pass type with no zeros (Ribeiro [2004, 2009]) given by:

\[
\frac{Z_1}{R} = \frac{X_1}{R} = G_u \left( \frac{1}{\omega_Ls + 1} \right) \left( \frac{1}{\omega_p^2 + \omega_{qu}^2 + \omega_{pu}^2} \right)
\]

(19) By comparison after expansion of the previous expression we easily determine the values for \( p_0, p_1, p_2 \) and \( p_3 \).

\[
p_3 = \omega_L G_u
\]

(20) \[p_2 = \frac{\omega_p^2}{Q_{pu}} + \omega_L\]

(21) \[p_1 = \omega_p^2 + \omega_L \frac{\omega_{qu}^2}{Q_{qu}}\]

(22) \[p_0 = \omega_L \omega_p^2\]

(23) To design the controller we select values for \( G_u, \omega_p, Q_{pu} \) and \( \omega_L \), then compute the values of \( p_0, p_1, p_3 \) and \( p_4 \) using the previous equations. Afterward, we apply those values to the pole placement feedback loop and in the end the system will behave as a linear loudspeaker with its frequency response adjusted according to the parameters one desires.

4. ADAPTATION AND PARAMETER ESTIMATION

We have determined a Feedback Linearization controller that will be able to linearize the loudspeaker. However, its control equation (15) heavily depends on the loudspeaker parameters, thus without good knowledge of their values the controller will have a limited capacity to reduce the loudspeaker distortion. Also notice the large number of parameters required. The pole placement control equation (16) is independent of any loudspeaker parameter but its operation depends on

3. SHAPING THE DESIRED FREQUENCY RESPONSE

The chain of integrators as in equation (9) does not yet give a suitable frequency response, inclusive it is unstable. Also, we are interested having the capability to specify the frequency response.
how well the feedback linearization controller is able to linearize the loudspeaker. Thus it is of great interest to make the controller adaptive.

The strategy followed here is to use a Model Reference Adaptive Control (MRAC) approach where the loudspeaker is always being controlled and its output is being compared to the output of a reference model. The controller parameters are directly updated on-line while the loudspeaker is operating normally. The resulting topology is shown in Figure 1.

The adaptive feedback linearization control will be similar to the one of equation (15) except for the use of estimates \( \hat{\theta}^1 \) and \( \hat{\theta}^2 \) of the parameter vectors instead of the real parameter vectors \( \theta^1 \) and \( \theta^2 \).

\[
u = \frac{1}{\theta^2 T \omega^2} (-\hat{\theta}^1 T \omega^1 + q) \quad (24)
\]

Using equations (7) and (24) in equation (7) we get an expression describing the loudspeaker behavior with regard to the parameter vectors \( \theta^1 \) and \( \theta^2 \).

\[
\dot{y} = \theta^1 T \omega^1 + \theta^2 T \omega^2 u \quad (25)
\]

Replacing \( u \) in equation (25) with equation (24) or, in other words, connecting the controller with the estimated parameters to the loudspeaker, after some manipulations and after defining the parameter errors \( \phi^1 = \theta^1 - \hat{\theta}^1 \) and \( \phi^2 = \theta^2 - \hat{\theta}^2 \) we arrive at the following expression

\[
\dot{y} = q + \phi^1 T \omega^1 + \phi^2 T \omega^2 u \quad (26)
\]

Now we apply the pole placement control law (equation (16)) to equation (26) using the fact that \( y = x, \dot{y} = v \) and \( \ddot{y} = \alpha \).

\[
\dot{y} = -p_0 y - p_1 \dot{y} - p_2 y + p_3 u + \phi^1 T \omega^1 + \phi^2 T \omega^2 u \quad (27)
\]

We need to define the reference system to which we compare the real system. This reference system can be defined by a state space model similar to equation (17)

\[
\dot{z}_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{bmatrix} z_m + \begin{bmatrix} 0 \\ 0 \\ p_3 \end{bmatrix} r \quad (28)
\]

with \( z_m = [x_m \ u_m \ a_m]^T \) as the state vector of the model. We already know how to specify the frequency response of this reference model system due to its similarity to the pole-placement case. To determine the values for \( p_0, p_1, p_2 \) and \( p_3 \) such that the model has a frequency response as specified by the values of \( G_u, \omega_p, Q_p \), and \( \omega_L \) we can again use equations (20) to (23).

Based on equation (28) the reference model can also be written as

\[
\dot{y}_m = -p_0 y_m - p_1 \dot{y}_m - p_2 y_m + p_3 r \quad (29)
\]

Note that we are assuming that both the model and the pole placement control have exactly the same parameters. This can be easily achieved because both are specified by design and independent of any loudspeaker parameters. Also, making the parameters different does not bring any advantage, so there is no apparent reason to do it.

So the adaptation error \( e = y - y_m \) can be determined by subtraction of equations (27) and (29).

\[
\dot{e} = -p_0 e - p_1 \ddot{e} - p_2 \dot{e} + \phi^1 T \omega^1 + \phi^2 T \omega^2 u \quad (30)
\]

which can be written in a matrix form

\[
\dot{e} = A_e e + b_1 (\phi^1 T \omega^1 + \phi^2 T \omega^2 u) \quad (31)
\]

where \( b_1 = [0 \ 0 \ 1]^T \), \( e = [e \ \dot{e} \ 2]^T \) and

\[
A_e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{bmatrix} \quad (32)
\]

The matrix \( A_e \) is directly imposed by the frequency response parameters \( p_0, p_1, p_2 \) selected in the design of the pole placement controller. This means that once the system’s response of the controlled system is set the error dynamics will remain fixed.

This last equation can also be written in a further simplified form as \( e = M(s) \omega^* \), where \( M(s) = c_1 (sI - A_e)^{-1} b_1 \) is the transfer function from \( \phi^1 T \omega^* \) to \( e \) and \( c_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \). Note that the \( M(s) \) transfer function, due to the use of pole placement, will always be identical to the desired transfer function for the whole system.

4.1 Finding the adaptation law

The key issue in the design of MRAC systems is the determination of the adaptive laws used to update the controller parameters. In this work we will use the Lyapunov direct method. According to the method, a system is asymptotically stable (in the sense of Lyapunov) at an equilibrium point \( x = 0 \) if a Lyapunov function \( V(x) \) exists satisfying:

\[
\begin{align*}
V(x) &> 0, \forall x \neq 0 \\
\dot{V}(x) &< 0, \forall x \neq 0 \\
V(x) &\to \infty, ||x|| \to \infty \\
V(0) &= 0
\end{align*}
\]

Fig. 1. MRAC using pole placement
The Lyapunov method consists in defining a positive function $V(x)$ and then determine a parameters update law such that its derivative $\dot{V}(x)$ will always be negative, thus making the system asymptotically stable. In the case of a MRAC, the system the state $x$ will correspond to a set of signals including the adaptation error vector $e$ and the controller parameters estimation error vector $\hat{e}$, so the system being asymptotically stable will mean that the both errors are limited and will always decrease.

Let us choose as Lyapunov function $V = e^T Pe + \phi^T \Gamma^{-1} \phi$, where the matrices $P$ and $\Gamma^{-1}$ are positive definite and symmetric. The adaptation gain matrix $\Gamma$ is chosen as a diagonal matrix with all elements positive, this way it is positive definite and $\Gamma^{-1}$ is also positive definite, $P$ is chosen to be symmetric. Differentiating $V$ we get

$$\dot{V} = e^T P \dot{e} + e^T \dot{P} e + 2\phi^T \Gamma^{-1} \dot{\phi}$$

Lets us replace $\dot{e}$ with the adaptation error expression (equation (31), both are the same).

$$\dot{V} = e^T (A_e^T P + PA_e) e + 2e^T Pb_1 \phi^T w^* + 2\phi^T \Gamma^{-1} \dot{\phi}$$

(33)

In order to guaranty that $\dot{V} < 0$ we need to have a matrix $Q$ definite positive and need to put the extra terms including $\phi$ to zero. The matrices $P$ and $Q$ can be selected with the help of the Lyapunov theorem stating that if the matrix $A$ has all its eigenvalues with strictly negative real parts, then, for any positive symmetric matrix $Q$ ($Q = Q^T > 0$) there is a positive symmetric matrix $P$ ($P = P^T > 0$) satisfying the Lyapunov equation:

$$A^T P + PA = -Q$$

(34)

Then, the first part of equation (33) simplifies to $-e^T Q e$ and the second part we equal to zero.

$$2e^T Pb_1 (\phi^T w^* + \phi^T w^2 u) + 2\phi^T \Gamma^{-1} \dot{\phi} = 0$$

After some algebraic manipulations we get

$$\dot{\phi} = -\Gamma (e^T Pb_1) w^*$$

The product $Pb_1$ is just a vector corresponding to the last column $p$ of the matrix $P$:

$$\dot{\phi} = -\Gamma (p^T e) w^*$$

Noting that $\dot{\phi} = \frac{d}{dt}(\theta - \hat{\theta}) = -\dot{\hat{\theta}}$ we get the adaptation law for the controller parameters $\hat{\theta}$:

$$\dot{\hat{\theta}} = +\Gamma (p^T e) w^*$$

(35)

The variable $\epsilon = e^T o^T$ is called the compensated error. The use of $pe^T$ makes the error equation SPR (Strictly Positive Real).

Equation (35) is the adaptation law and requires the knowledge of all elements of $e$ that usually are not available because only the output of the process is available and not its internal state.

In general, the adaptation law is something like

$$\dot{\hat{\theta}} = +\Gamma \epsilon\xi$$

(36)

where $\epsilon$ is directly related to $e$ and $\xi$ may be the vector $w^*$ itself or a filtered version of it. These modifications are required to make sure that the linear part of the equation error is SPR ($H(j\omega) > 0 \forall \omega \geq 0$, or number of poles and zeros differs at most 1, or phase shift is never larger than 90°).

The error equation (equation (31)) and the adaptation law (equation (36)) together form a feedback loop as represented in figure 2 where $z = -\phi^T \xi$. The upper part is the linear part (thus time invariant) of the error dynamics and its state vector contains all of the error dynamics states. The lower part is the nonlinear part of the error dynamics, is time-varying due to the input $\xi$ and is memory-less, meaning that it has no internal states. Both parts are interconnected in a feedback loop like configuration. It is known that a feedback connection like this one is asymptotically stable at the origin if the upper part is strictly passive and the lower part is passive.

The strict passivity of the upper part is guaranteed if we make its transfer function strictly positive real (SPR)Butler [1992], Brogliato et al. [2007]. If all elements of $e$ are accessible, this can always be accomplished by proper selection of the elements of the vector $p$ (recall that the only requirement for $p$ is to be the last column of a symmetric positive definite matrix $P$).

Now suppose we will use the acceleration $a$ as the output of the loudspeaker to compare with the reference model. In that case, the adaptation error will be $e_a = a - a_m = \ddot{y} - \ddot{y}_m = \ddot{e}$. The error equation remains the same $\ddot{e} = A_e e + b_1 \phi^T w^*$ but now we do not have the whole state $e$ available, only $\ddot{e}$. This means we have a output vector $b_\epsilon$ such that $e_\epsilon = b_\epsilon e = \left[ 0 \ 0 \ 1 \right]^T e$. In this case, the error equation remains SPR. Then we can make $\epsilon = e_a$ and $\xi = w^*$ and the adaptation law is simply

$$\dot{\hat{\theta}} = -\dot{\phi} = +\Gamma e_a w^*$$

(37)
In other cases, for example when comparing the position $x$ instead of the acceleration $a$, the error equation ceases to be SPR. In that case, the so called “Augmented Error” needs to be use. This will substantially increase the complexity of the adaptive algorithm and, for that reason and because another solution (the use of the acceleration $a$) was found, we will not discussed the augmented error here. The interested reader should consult Butler [1992], Ribeiro [2009].

5. SIMULATION

Software using the C++ computer language was written to allow for the simulation of the adaptive controllers discussed in this report. As an example, the results from a simulation are presented in figure (3). The loudspeaker simulated was controlled by a feedback linearization controller using pole placement. Initially, the controller was initialized with the parameters that where correct for the loudspeaker in question. Then, at $t = 0.5$ s, the parameters where reset zero, increasing the adaptation error (depicted) substantially as as expected. Afterward (at $t = 1$ s) the adaptation was enabled an we can clearly see the adaptation error diminishing down to the initial error level. This means that the adaptive algorithm is being successfully at lowering the error, as was desirable.

6. CONCLUSION

A Controller based on the application of the feedback linearization technique was designed in order to reduce the nonlinear effects of the electrodynamic loudspeaker. By the introduction of an approximated polynomial $L_{lin}(x_1) \approx \frac{1}{L(x_1)}$ we where able to design the controller in a way so that it can be represented by a product of two parameter vectors by two signal vectors, making it possible the application of an adaptive algorithm to the controller. Afterwards, pole placement was applied in order to compensate the frequency response of the controlled loudspeaker. For the adaptation, the Model Reference Adaptive Control strategy was used so that the controller gets its parameters directly updated without further computations required and while the system is online.

Some simulations where performed showing the correct functioning of the nonlinear adaptive controller designed in this work.

References


