ADAPTIVE FEEDBACK LINEARIZING CONTROL FOR TRANSPORT PHENOMENA PROCESSES

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Abstract: Adaptive feedback linearizing control of distributed plants involving transport phenomena, described by partial differential equations is considered. The method proposed relies on Lyapunov’s stability to obtain parameter estimates and tackles directly infinite dimension systems without finite dimension approximations. The control of a tubular counter-current heat exchanger is presented as an example to illustrate the method.

Keywords: Nonlinear Systems, Partial Differential Equations, Adaptive Control, Linearizing Feedback

1. INTRODUCTION

This work proposes a systematic approach to the adaptive control of a class of nonlinear systems described by partial differential equations based on feedback linearization and Lyapunov’s stability. This contribution follows the work of other researchers [7], [6].

The theory of linearization by exact feedback for continuous finite dimension systems was developed through the efforts of several authors in the last twenty years [10], [9]. The main drawback seems to be in the fact that it relies on exact cancellation of nonlinear terms to yield a linear input-output behavior. Consequently if model errors occur the cancellation is not exact and the input-output dynamics is no longer linear. Adaptive control techniques combined with feedback linearization can improve robustness and performance under parametric uncertainty [17], [2].

The paper is organized as follows: after this introduction, in section 2, the prototype model of a class of tubular systems, modelled by a set of hyperbolic partial differential equations (PDE), is described. Section 3 deals with a systematic approach to find adaptive control laws based in linearizing feedback for the prototype model. In section 4 an application of these techniques to a counter-current heat exchanger is presented. Section 5 draws some conclusions.

2. MODELS FOR TRANSPORT PHENOMENA SYSTEMS

The processes considered evolves in simple or composite tubular geometry, as shown in fig. 1. In the first case the process is confined to the interior of a cylindrical metallic pipe where the mixture, assumed single-phase, moves at velocity $v(t)$. For the composite geometry, in addition to the inner pipe, there is a concentric outer pipe that allows to transfer heat for the mixture that flows in the inner pipe. The fluid in the outer pipe may be in co-current or counter-current with respect to the fluid in the inner pipe. These two prototype configurations allow to study a wide variety of processes with transport phenomena [3] including tubular reactors [14] and bio-reactors [8], heat exchangers and solar fields [5] and they form the basis for the study of systems with more complex geometries.

In this paper it is assumed that the control $u$ is made by manipulating the fluid velocity in the case of simple geometry and the velocity of the outer fluid in the
variable is the transport velocity \( u \) resulting from applying conservation principles (mass hyperbolic PDEs in the state variables is the pipe length and \( u \) by manipulating the fluid velocity.

\[ \text{The objective is to control the output, a smooth non-} \]

\[ \text{linear state function weighted over space:} \]

\[ g(t) = \int_0^1 p(z)h(x(z,t)) \, dz; \int_0^1 p(z) \, dz = 1 \]

by manipulating the fluid velocity \( u(t) \).

3. ADAPTIVE FEEDBACK LINEARIZING CONTROL

The subject of this section is to design adaptive controllers based on feedback linearizing control combined with update laws for estimation of uncertain parameters. The adaptation laws are guaranteed stable and are obtained based on Lyapunov’s direct method [13], [18]. Using Lyapunov’s method in adaptive design, the adaptive law is directly obtained from the stability condition, in the following way: first a usable error equation is derived; second, a Lyapunov function is chosen as a function of both signal error and parameter error; third, the time derivative of the Lyapunov function is calculated and is guaranteed to be negative defined by putting extra terms, including the parameter error, to zero [4]. Putting the extra terms to zero provides the adaptive law.

3.1 Adaptive Control without observer

Consider a plug-flow tubular system with simple geometry. In this case the hyperbolic PDE equation can be written in the following matricial form:

\[ \frac{\partial x}{\partial t} + u \frac{\partial x}{\partial z} = f(x; \theta) \]

\[ y = \int_0^1 p(z)h(x) \, dz \]

with:

\[ f(x; \theta) = f_0(x) + \sum_{i=1}^q \theta_i f_i(x) = f_0(x) + F(x)\theta \]

where \( A_v \) is a diagonal matrix with one or zero main diagonal elements, depending if there is a related transport term or not, \( f_0(x) \) is a vector of smooth nonlinear functions \( (n \times 1) \), \( F(x) \) is a matrix of smooth nonlinear functions \( (n \times q) \) and \( \theta = [\theta_1 \cdots \theta_q]^T \) is an uncertain parameters vector assumed time constant. Assuming minimal phase, using linearizing feedback, and nothing that the relative degree is one, it yields [6], [10]:

\[ u = -v + \int_0^1 p(z) \frac{\partial h}{\partial z} f_0(x) \, dz \]

\[ + \int_0^1 p(z) \frac{\partial h}{\partial z} F(x) \, dz \hat{\theta} \]

with \( \int_0^1 p(z) \frac{\partial h}{\partial z} A_v \frac{\partial \hat{\theta}}{\partial z} \, dz \neq 0 \) in the neighborhood of an equilibrium point, \( v \) is a virtual input and \( \hat{\theta} \) is replaced by its estimated value \( \hat{\theta} \) (certainty equivalence principle).

The input-output dynamics is no longer just an integrator, it also includes a disturbance term proportional to the parameter estimation error \( \hat{\theta} \).
\[ \dot{y} = v + \Lambda_f \tilde{\theta} \quad (7) \]

with \( \tilde{\theta} = \theta - \hat{\theta} \) and:

\[ \Lambda_f = \int_0^1 p(z) \frac{\partial h}{\partial x} F(x) \, dz \quad (8) \]

For:

\[ v = \dot{r} + \gamma_1 (r - y) \quad (9) \]

with \( \gamma_1 > 0 \), the error dynamics, \( e = r - y \), is given by:

\[ \dot{e} + \gamma_1 e + \Lambda_f \dot{\theta} = 0 \quad (10) \]

A stable adaptation law can be established through the following Lyapunov candidate function:

\[ V(e, \dot{\theta}) = \frac{1}{2} \left( e^2 + \rho^{-1} \dot{\theta}^T \dot{\theta} \right) \quad (11) \]

where \( \rho \) is a real positive parameter. The time derivative is:

\[ \dot{V} = e \dot{e} + \rho^{-1} \dot{\theta}^T \dot{\theta} = -e \left( \gamma_1 e + \Lambda_f \dot{\theta} \right) + \rho^{-1} \dot{\theta}^T \dot{\theta} \quad (12) \]

or:

\[ \dot{V} = -\gamma_1 e^2 + \dot{\theta}^T \left( -\Lambda_f e + \rho^{-1} \dot{\theta} \right) \quad (13) \]

Choosing:

\[ \dot{\theta} = \rho \Lambda_f e \quad (14) \]

or:

\[ \dot{\theta} = -\rho \Lambda_f e \quad (15) \]

That implies:

\[ \dot{V} (e) = -\gamma_1 e^2 \quad (16) \]

note that:

\[ \dot{V} (e) < 0 \quad e \neq 0 \quad (17) \]

Consequently, under (6) and (15) the closed-loop system:

\[
\frac{\partial x}{\partial t} + \left[ \frac{\partial H}{\partial x} + \frac{1}{\rho^2} \frac{\partial^2 h}{\partial x^2} F(x) \right] \frac{\partial h}{\partial x} \frac{\partial^2 x}{\partial x^2} + \int_0^1 p(z) \frac{\partial^2 h}{\partial x^2} F(z) \frac{\partial x}{\partial z} A_v \frac{\partial \hat{x}}{\partial z} + \int_0^1 p(z) \frac{\partial^2 h}{\partial x^2} F(z) \frac{\partial x}{\partial z} A_v \frac{\partial \hat{x}}{\partial z} = f(x; \theta) \\
\dot{\hat{\theta}} = -\rho \Lambda_f^T e 
\]  

is stable, assuming that the null dynamics is stable and the denominator \( \int_0^1 p(z) \frac{\partial^2 h}{\partial x^2} A_v \frac{\partial \hat{x}}{\partial z} \neq 0 \) locally. The null dynamics is obtained with \( \dot{r} = 0 \) and \( r \) constant. If null dynamics is stable the system is called minimal phase [10].

Note that this transformation renders the state dynamics \( x(z, t) \) not observable as the estimative error becomes zero, except for the tracking error mode, \( (r(0) - y(0))e^{-\gamma_1 t} \).

Note also that the stability condition guarantees the tracking error convergence to zero, but does not assure zero convergence of the estimative error. For that it is necessary to show that the invariant set for \( (e, \dot{\theta}) \) is the origin which, in this case, means that \( \Lambda_f \neq 0 \) locally. Technically, to demonstrate the asymptotic stability for the trajectories its needed to evoke LaSalle’s Invariant Set Theorem [13], [18]. According to this theorem, all trajectories approach the major invariant set contained in the set defined by \( V(e) = 0 \). By (16), the invariant set is not empty including \( e = 0 \) and so all the trajectories verifies \( e \rightarrow 0 \) as \( t \rightarrow \infty \). If the two sets coincides with the origin then also \( \theta \rightarrow 0 \) as \( t \rightarrow \infty \).

Consider now the same simple geometry with dispersion model (parabolic PDE):

\[ -\frac{D}{L^2} \frac{\partial^2 x}{\partial z^2} + \frac{u}{L} A_v \frac{\partial x}{\partial z} + \frac{\partial x}{\partial t} = f(x; \theta) \]

\[ y = \int_0^1 p(z) h(x) \, dz \quad (19) \]

Using again feedback linearizing, the control law is given by:

\[ u = \frac{-v + \int_0^1 p(z) \frac{\partial h}{\partial z} A_v \frac{\partial^2 x}{\partial z^2} \, dz}{\int_0^1 p(z) \frac{\partial h}{\partial z} A_v \frac{\partial^2 x}{\partial z^2} \, dz} \]

\[ + \int_0^1 p(z) \frac{\partial h}{\partial z} f_0(x) \, dz + \int_0^1 p(z) \frac{\partial h}{\partial z} F(x) \, dz \frac{\partial \hat{x}}{\partial t} \]

\[ \frac{\partial x}{\partial t} = \frac{-r - \gamma_1 r + \int_0^1 p(z) \frac{\partial h}{\partial z} \left( \gamma_1 h(x) + \frac{\partial h}{\partial z} f_0(x) \right) \, dz}{\int_0^1 p(z) \frac{\partial h}{\partial z} A_v \frac{\partial^2 x}{\partial z^2} \, dz} \]

\[ + \int_0^1 p(z) \frac{\partial h}{\partial z} F(x) \, dz \frac{\partial \hat{x}}{\partial z} + \int_0^1 p(z) \frac{\partial h}{\partial z} F(x) \, dz \frac{\partial \hat{x}}{\partial z} + \frac{\partial x}{\partial z} = f(x; \theta) \\
\hat{\theta} = -\rho \Lambda_f^T e \]

Again if the system is minimum phase, closed-loop stability is guaranteed and using the previous arguments asymptotic stability can be checked.

### 3.2 Adaptive Control with observer

The techniques used in the previous section, for systems of simple geometry, have structural problems when applied to composite geometry systems for which the relative degree is two. In this case it is not possible to write an explicit tracking error equation, as in the previous case, since this will depend on \( \dot{y} \) which in turn depends on the parameters and so it is not possible to isolate the term in \( \dot{\theta} \). An alternative is
to use an observer to the parameters since the state is considered full accessible.

Consider model (2) in the following matricial format:

\[
\frac{\partial y}{\partial t} = \begin{bmatrix}
    f(\eta) + \frac{p(\eta, d)}{L^2} \frac{\partial^2 x}{\partial z^2} - \frac{A_x \partial x}{L} \\
    0 \frac{\partial^2 x}{\partial z^2}
\end{bmatrix} + \begin{bmatrix}
    \frac{\partial w}{\partial z} \\
    0
\end{bmatrix} \begin{bmatrix}
u \\
    L
\end{bmatrix}
\]

where \( \eta = [w \ x^T]^T \) and output:

\[
y = \int_0^1 p(z) \ h(\eta(z, t)) \ dz
\]

Consider also that the output does not explicitly depends on \( w : \frac{\partial h}{\partial \eta} = 0 \) and:

\[
f(\eta; \theta) = f_0(\eta) + \sum_{i=1}^q \theta_i f_i(\eta) = f_0(\eta) + F(\eta) \theta
\]

\[
p(\eta; \theta) = p_0(\eta) + \sum_{i=1}^q \theta_i p_i(\eta) = p_0(\eta) + P(\eta) \theta
\]

which allows to write:

\[
\frac{\partial \eta}{\partial t} = \begin{bmatrix} \mathcal{L} \end{bmatrix} \pm \begin{bmatrix}
\frac{\partial w}{\partial z} \\
0 \frac{\partial^2 x}{\partial z^2}
\end{bmatrix} \begin{bmatrix}
u \\
L
\end{bmatrix} + \begin{bmatrix}
p_0(\eta) \\
0
\end{bmatrix} + \begin{bmatrix} P(\eta) \\
0
\end{bmatrix} \theta
\]

where:

\[
\mathcal{L} = \frac{D}{L^2} \frac{\partial^2 x}{\partial z^2} - \frac{A_x \partial x}{L}
\]

Writing the observer filter dynamics in the form [19], [1]:

\[
\frac{\partial \eta_o}{\partial t} = \begin{bmatrix} \mathcal{L} \end{bmatrix} \pm \begin{bmatrix}
\frac{\partial w}{\partial z} \\
0 \frac{\partial^2 x}{\partial z^2}
\end{bmatrix} \begin{bmatrix}
u \\
L
\end{bmatrix} + \begin{bmatrix} p_0(\eta) \\
0
\end{bmatrix} + \begin{bmatrix} P(\eta) \\
0
\end{bmatrix} \theta + \mathcal{K} (\eta - \eta_o)
\]

where \( \mathcal{K} > 0 \), the observation error dynamics is given by:

\[
\frac{\partial e}{\partial t} = \begin{bmatrix} p(\eta) \\
F(\eta)
\end{bmatrix} \dot{\theta} - \mathcal{K} e
\]

Consider the Lyapunov candidate function:

\[
V(e, \dot{\theta}) = \frac{1}{2} \int_0^1 e^T e \ dz + \dot{\theta}^T \Gamma^{-1} \dot{\theta}
\]

Differentiating \( V \) with respect to time, one obtains:

\[
\dot{V} = \frac{1}{2} \int_0^1 \left( \frac{\partial e}{\partial t} \right)^T e + e^T \frac{\partial e}{\partial t} \right) \ dz + \frac{1}{2} \left( \dot{\theta}^T \Gamma^{-1} \dot{\theta} + \dot{\theta}^T \Gamma^{-1} \dot{\theta} \right)
\]

and using the error dynamics it yields:

\[
\dot{V} = \int_0^1 e^T \mathcal{K} e \ dz + \dot{\theta}^T \left( \int_0^1 \begin{bmatrix} p(\eta) \\
F(\eta)
\end{bmatrix} e \ dz + \Gamma^{-1} \dot{\theta} \right)
\]

Choosing :

\[
\dot{\theta} = -\Gamma \int_0^1 \begin{bmatrix} p(\eta) \\
F(\eta)
\end{bmatrix} e \ dz
\]

guarantees:

\[
\dot{V}(e) \leq 0; \ e_o \neq 0
\]

and so, \( e_o(t) \to 0 \) as \( t \to \infty \).

When (33) is combined with the feedback linearizing input law:

\[
u = \mp v \pm \int_0^1 p(z) \left( \frac{\partial^2 x}{\partial z^2} \dot{\theta} - \frac{\partial^2 x}{\partial z^2} \dot{\theta} \right) \ dz
\]

where:

\[
\dot{x}(\eta, \dot{\theta}) = \begin{bmatrix} p(\eta, \dot{\theta}) \\
F(\eta, \dot{\theta}) + \mathcal{L} x
\end{bmatrix}
\]

\[
v = \gamma_1 (r - y) + \gamma_2 (\dot{r} - \dot{y}) + \ddot{r}
\]

\[
\dot{y}(\eta, \dot{\theta}) = \int_0^1 p(z) \frac{\partial h}{\partial \eta} \ dz
\]

the closed-loop is stable if the system is minimum phase and, by the Invariant Set Theorem, the errors go to zero after initial conditions transient \( w(0), x(0), \dot{\theta}(0) \), if:

- \( u \) bounded or its denominator bounded away from zero, that can imply using a parameters projection [12]:

\[
\dot{\theta} = \text{Proj} \left( \Gamma \int_0^1 \begin{bmatrix} p(\eta) \\
F(\eta)
\end{bmatrix} e \ dz \right)
\]

- the nonlinear functions matrix \( \begin{bmatrix} p(\eta) \\
F(\eta)
\end{bmatrix} \neq 0 \) locally as a way to assure that the adaptation error goes to zero.

Closed-loop stability is then guaranteed and asymptotic stability can be checked.
Consider the model of a counter-current heat exchanger:

\[
\frac{\partial w}{\partial t} - \frac{w}{L} \frac{\partial w}{\partial z} = a(w - x) \tag{42}
\]

\[
\frac{\partial x}{\partial t} + v \frac{\partial x}{\partial z} = b(x - w) \tag{43}
\]

in which \(w\) and \(x\) are respectively the outer and inner pipe fluid temperature, \(a\) and \(b\) the outer and inner exchange coefficient \([\text{min}^{-1}]\), \(v\) is the inner pipe fluid velocity \([\text{m min}^{-1}]\) and the manipulated variable is the outer pipe fluid velocity \(u\). The output is given by:

\[
y(t) = \int_0^1 x(z, t) \, dz \tag{44}
\]

which means the output is proportional to the cold liquid total energy \([11]\):

\[
U(t) = \rho C_p A \int_0^1 x(z, t) \, dz \tag{45}
\]

where \(\rho\) is the fluid density, \(C_p\) is the specific heat and \(A\) is the pipe section.

The control law, in this case, is obtained differentiating twice the output with respect to time, allowing to obtain:

\[
u = \frac{v - (a^2 + ab)(y - \bar{w}) - \nu a \Delta w}{\bar{w}} + v \frac{d}{dt} \frac{\Delta x}{L} \tag{46}
\]

where:

\[
\bar{w} = \int_0^1 w(z, t) \, dz \tag{47}
\]

\[
\Delta x = \int_0^1 \frac{\partial x}{\partial z} \, dz = x(1, t) - x(0, t) \tag{48}
\]

\[
\Delta w = \int_0^1 \frac{\partial w}{\partial z} \, dz = w(1, t) - w(0, t) \tag{49}
\]

\[
\frac{d}{dt} \frac{\Delta x}{L} = \frac{\dot{x}(1, t) - \dot{x}(0, t)}{L} \tag{50}
\]

with:

\[
\dot{x}(1, t) = -v \frac{\partial x}{\partial z} \big|_{z=1} + b(x(1, t) - w(1, t)) \tag{51}
\]

and also:

\[
v = \dot{y} = \gamma_0 \int_0^1 (r - y) \, d\tau + \gamma_1 (r - y) + \gamma_2 (\dot{r} - \dot{y}) + \dot{\bar{w}} \tag{52}
\]

where \(v\) is a virtual input. Its assumed that \(\gamma_0 + \gamma_1 s + \gamma_2 s^2 + s^3\) is Hurwitz polynomial and also that the null dynamics is exponentially stable. For the control law to be well defined \(\Delta w \neq 0\) for all \(t\).

In the adaptive case and assuming the certain equivalence, the parameters are replaced by estimates:

\[
\hat{a} = \dot{\hat{a}} = \frac{\Delta w}{\Delta x} \tag{53}
\]

\[
\hat{b} = \dot{\hat{b}} = \frac{\Delta w}{\Delta x} \tag{54}
\]

\[
\hat{\gamma}_0 = \gamma_0 \int_r^1 (r - y) \, d\tau + \gamma_1 (r - y) + \gamma_2 (\dot{r} - \dot{y}) + \dot{\bar{w}} \tag{55}
\]

\[
\hat{\gamma}_1 = \gamma_1 \tag{56}
\]

Table 1. Heat Exchanger parameters, initial and boundary conditions.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.2</td>
<td>(\text{min}^{-1})</td>
</tr>
<tr>
<td>(b)</td>
<td>0.2</td>
<td>(\text{min}^{-1})</td>
</tr>
<tr>
<td>(w(0, 0))</td>
<td>50.0</td>
<td>(^\circ\text{C})</td>
</tr>
<tr>
<td>(x(0, 0))</td>
<td>50.0</td>
<td>(^\circ\text{C})</td>
</tr>
<tr>
<td>(x(0, 1))</td>
<td>25.0</td>
<td>(^\circ\text{C})</td>
</tr>
</tbody>
</table>

with local observation dynamics, respectively:

\[
\dot{\eta}_a = \frac{v}{L} \left. \frac{\partial x}{\partial z} \right|_{z=1} + \hat{a} (w(1, t) - x(1, t)) \tag{57}
\]

\[
\dot{\eta}_b = \frac{v}{L} \left. \frac{\partial x}{\partial z} \right|_{z=1} + \hat{b} (x(0, t) - w(0, t)) \tag{58}
\]

In this case, it is necessary to project the parameters, because physically \(a, b > 0\), and to get the control law with the certain equivalence well defined \(\Delta w > 0\) for all \(t\).

The parameters projection is done by using the following rule [12]:

\[
\text{Proj} \{ \tau, (\bar{\theta}, \bar{\bar{\theta}}, \epsilon) \} =
\begin{cases}
\max(0, \frac{\epsilon - \hat{\theta} + \bar{\theta}}{\epsilon}) \tau \hat{\theta} \geq \bar{\theta} \epsilon \tau > 0 \\
\max(0, \frac{\epsilon + \hat{\theta} - \bar{\theta}}{\epsilon}) \tau \bar{\theta} \leq \hat{\theta} \epsilon \tau < 0 \\
\text{otherwise}
\end{cases}
\]

with \(\bar{\theta} + \epsilon \geq \hat{\theta} \geq \bar{\theta} - \epsilon > 0\) where \(\tau\) is the update law.

Figs. 2-5 show closed-loop behavior for a 35\(^\circ\text{C}\) final value step set point, \(a = b = 0.2\) and \(\hat{\theta}(0) = \bar{\theta}(0) = 0.5\). Figs. 6-9 show closed-loop behavior for a 20% change in parameter \(a\), meaning \(a(t > 120) = 0.25\). This simulation uses 200 space elements and values from tabs. 1 and 2.
Table 2. Controller parameters.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
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<tr>
<td>γ₀</td>
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<tr>
<td>γ₁</td>
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<td>κₐ</td>
<td>0.500</td>
</tr>
<tr>
<td>κₜ</td>
<td>0.500</td>
</tr>
</tbody>
</table>

This example shows most of the positive aspects of combining feedback linearizing control with adaptive Lyapunov laws to control parameter uncertain nonlinear infinite dimension systems. In this case, the use of space local observers avoid the need of full state access. Full state access is only require by the energy output used.

5. CONCLUSIONS

An adaptive linearizing feedback control for processes with transport phenomena, in which the manipulated variable is the velocity, has been systematically developed for the prototype system. Adaptive laws for the parameters, stable in the Lyapunov’s sense, were obtained. In the simple geometry system case it was possible to write a useful tracking error equation to use Lyapunov’s method because the system’s relative degree is one. For the composite system it was not possible to write a direct error equation, as in the previous case. The alternative was to use an observer to the parameters, since full state accessibility is considered.
Closed-loop stability was established in both cases. A numerical example illustrates the method.

6. REFERENCES